

Problem Set 3

Phys 7240
Due: Feb 17

1 Correlation Functions

The action for a harmonic oscillator (in imaginary time) is given by

$$S = \frac{m}{2} \int d\tau [\dot{x}^2 + \omega^2 x^2] = \frac{m}{2} \int d\tau \left\{ x \left[-\frac{d^2}{d\tau^2} + \omega^2 \right] x \right\}. \quad (1.1)$$

The correlation function $D(t)$ for a harmonic oscillator can be defined in two equivalent ways

$$D(t) = \langle 0 | T \hat{x}(t) \hat{x}(0) | 0 \rangle = \frac{1}{\int \mathcal{D}x} \int \mathcal{D}x x(0) x(t) e^{-S}. \quad (1.2)$$

Here $\hat{x}(t)$ is the coordinate operator in the Heisenberg representation $x(t) = e^{\hat{H}t} x e^{-\hat{H}t}$, and T stands for time ordering. The correlation function can easily be calculated if one uses that the integral (even a functional integral) over a total derivative is zero.

$$0 = \int \mathcal{D}x \frac{\delta}{\delta x(t)} [x(0) e^{-S}] = \delta(t) \int \mathcal{D}x e^{-S} - m \left[-\frac{d^2}{dt^2} + \omega^2 \right] \int \mathcal{D}x x(t) x(0) e^{-S}. \quad (1.3)$$

Consequently,

$$m \left[-\frac{d^2}{dt^2} + \omega^2 \right] D(t) = \delta(t). \quad (1.4)$$

1. Calculate $D(t)$ by solving Eq. (1.4).
2. Calculate $D(t)$ in an alternative way by using

$$D(t) = \langle 0 | T \hat{x}(t) \hat{x}(0) | 0 \rangle = \sum_n |\langle n | \hat{x}(0) | 0 \rangle|^2 e^{-(E_n - E_0)t} \quad (1.5)$$

3. Knowing the correlation functions allows one to deduce the commutation relations between the operators. For example,

$$\lim_{\epsilon \rightarrow 0} [\dot{D}(\epsilon) - \dot{D}(-\epsilon)] = \langle 0 | \hat{x}(0) \hat{x}(0) - \hat{x}(0) \hat{x}(0) | 0 \rangle. \quad (1.6)$$

Use $D(t)$ you calculated above to figure out the commutation between the operators \hat{x} and \hat{p} . Using the relation between the momentum $\hat{p} = -i \frac{d}{dx}$ and \hat{x} , check that you indeed reproduce the commutation relation between the coordinate and the momentum.

2 Easy Plane Magnet in a Transverse Field

The easy plane magnet in a transverse field is defined as

$$H = -\gamma \sum_x \tau_x^3 - \beta \sum_x \left(\tau_x^1 \tau_{x+1}^1 + \tau_x^2 \tau_{x+1}^2 \right). \quad (2.1)$$

This describes a chain of spins- $\frac{1}{2}$ which prefer to lie in the $x-y$ plane, subject to a magnetic field in the z -direction.

1. Map this into a free fermion problem using the Jordan-Wigner transformation.
2. Find the ground state energy and the correlation length.

3 Bose-Einstein condensation in a weakly interacting gas

A weakly interacting boson gas at zero temperature is well described by the following Hamiltonian

$$H = \sum_k \left(\frac{k^2}{2m} + 2\Delta \right) a_k^\dagger a_k + \Delta \sum_k \left(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger \right). \quad (3.1)$$

Here a_k, a_k^\dagger are bosonic annihilation and creation operators which satisfy

$$a_k a_k^\dagger - a_k^\dagger a_k = 1, \quad (3.2)$$

and Δ is related to the density of the Bose-Einstein condensate. For more information about this Hamiltonian, you could see, for example, Landau-Lifshitz, volume 9, chapter on superfluidity.

1. Bring this Hamiltonian to the matrix form

$$H = \sum_k \begin{pmatrix} a_k^\dagger & a_{-k}^\dagger & a_k & a_{-k} \end{pmatrix} \mathcal{H}_k \begin{pmatrix} a_k \\ a_{-k}^\dagger \\ a_k \\ a_{-k}^\dagger \end{pmatrix} \equiv \sum_k c_k^\dagger \mathcal{H}_k c_k. \quad (3.3)$$

2. Find the conditions a matrix U must satisfy so that $d = Uc$ would still satisfy the bosonic commutation relations Eq. (3.2).
3. Diagonalize the matrix \mathcal{H}_k with the help of the transformation defined by U and find the spectrum of the Hamiltonian Eq. (3.1). This is the spectrum of phonon excitations in a weakly interacting BEC.