Problem Set 2

Phys 7240
Due: Feb 3

1 Free Particle

The Hamiltonian of a free particle in 1D is given by

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}. \tag{1.1}$$

1. Compute its Green's function by using the formula

$$G(x, x_0; t) = \sum_n e^{-iE_n t}\psi_n(x)\psi_n^*(x_0) \tag{1.2}$$

2. Check that

$$i\frac{\partial G}{\partial t} = -\frac{1}{2m}\frac{\partial^2 G}{\partial x^2},\ \ t > 0, \tag{1.3}$$

that

$$G(x, x_0; 0) = \delta(x - x_0), \tag{1.4}$$

and that

$$G(x, x_0, t_1 + t_2) = \int_{-\infty}^{\infty} dy\ G(x, y, t_1)G(y, x_0, t_2). \tag{1.5}$$

3. Check that

$$G(x, x_0, N\epsilon) = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \int_{-\infty}^{\infty} \prod_{l=1}^{N-1} dx_l\ e^{i\sum_{i=0}^{N-1} m\frac{(x_{i+1}-x_i)^2}{2\epsilon}} \equiv \int \mathcal{D}x(t)e^{iS}. \tag{1.6}$$

2 Harmonic Oscillator

The action for a harmonic oscillator is given by

$$S = \frac{1}{2}\int_0^T dt\ \left[m\dot{x}^2 - \omega_0^2 x^2\right] = \frac{1}{2}\int_0^T dt\ \left\{x\left[-m\frac{d^2}{dt^2} - \omega_0^2\right]x\right\}. \tag{2.1}$$

In this problem you will calculate the Green's function of the harmonic oscillator between the points $x(0) = 0$ and $x(T) = 0$, $G(0, 0; T)$, using the approach of operator determinants.

$$G(0, 0; T) \propto \frac{1}{\sqrt{\det\left(-m\frac{d^2}{dt^2} - \omega_0^2\right)}}. \tag{2.2}$$

1. Find the eigenvalues λ_n of the operator $-m\frac{d^2}{dt^2} - \omega_0^2$ by solving

$$\left[-m\frac{d^2}{dt^2} - \omega_0^2\right]\psi_n(t) = \lambda_n\psi_n(t) \tag{2.3}$$

with the boundary conditions $\psi(0) = 0$, $\psi(T) = 0$.

The product $\prod_n \lambda_n$ is divergent. The reason for that is that the factor $\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}$ from the path integral definition is also divergent in the $\epsilon \to 0$, $N \to \infty$ limit in such a way as to compensate the divergence of the determinant. Although it is difficult to take this factor into account, it is possible to avoid dealing with it altogether by introducing $G_0(0, 0; T)$, the Green's function of a free particle from Problem 1, and writing

$$\frac{G(0,0;T)}{G_0(0,0;T)} = \sqrt{\frac{\det\left(-m\frac{d^2}{dt^2}\right)}{\det\left(-m\frac{d^2}{dt^2} - \omega_0^2\right)}} = \sqrt{\prod_n \frac{\lambda_n^0}{\lambda_n}}, \tag{2.4}$$

λ_n^0 are the eigenvalues of the operator $-m\frac{d^2}{dt^2}$, or in other words, they are λ_n if $\omega_0 = 0$. The product in Eq. (2.4) is now convergent.

2. Evaluate the product in Eq. (2.4). Use your knowledge of the Green's function of a free particle from Problem 1 to find $G(0, 0; T)$.

3. Compare the answer with the formula

$$G(x, x_0; t) = \sum_n e^{-iE_nt}\psi_n(x)\psi_n^*(x_0). \tag{2.5}$$

What does it tell you about the energy levels E_n of the harmonic oscillator?

3 1D Ising model in a magnetic field

The 1D Ising model in a magnetic field is defined by its energy

$$E = -J\sum_{k=0}^{N-1}\sigma_k\sigma_{k+1} - h\sum_{k=0}^{N}\sigma_k \tag{3.1}$$

The <u>connected</u> correlation function and the correlation length ξ of the Ising model is defined as

$$\langle\,\sigma_l\sigma_n\,\rangle - \langle\,\sigma_l\,\rangle\langle\,\sigma_n\,\rangle \propto e^{-\frac{|l-n|}{\xi}}, \tag{3.2}$$

where

$$\langle\,\sigma_l\,\rangle = \frac{\sum_{\{\sigma\}=\pm1}\sigma_l e^{-\frac{E}{T}}}{\sum_{\{\sigma\}=\pm1} e^{-\frac{E}{T}}},\ \langle\,\sigma_l\sigma_n\,\rangle = \frac{\sum_{\{\sigma\}=\pm1}\sigma_l\sigma_n e^{-\frac{E}{T}}}{\sum_{\{\sigma\}=\pm1} e^{-\frac{E}{T}}}. \tag{3.3}$$

Find the correlation length ξ using the transfer matrix/Hamiltonian formalism.