Final Exam

Phys 7240
Dec 7

Transfer Matrix Formalism

Consider the 3-state Potts model in 1D with periodic boundary conditions, which is defined as

$$Z = \sum_{\sigma_i=1,2,3} e^{J\sum_{k=1}^{N} \delta_{\sigma_k,\sigma_{k+1}}}.$$

Here $\sigma_{N+1} \equiv \sigma_1$ and $\delta_{\sigma,\sigma'}$ is equal to 1 if $\sigma = \sigma'$ and to 0 if $\sigma \neq \sigma'$.

1. Write down its transfer matrix.

Solution: The transfer matrix is

$$T_{\sigma,\sigma'} = e^{J\delta_{\sigma\sigma'}} = \begin{pmatrix} e^J & 1 & 1 \\ 1 & e^J & 1 \\ 1 & 1 & e^J \end{pmatrix}$$

Notice that $e^0 = 1$, and not zero!

2. Compute this partition function (in the limit $N \to \infty$).

Solution:

$$Z = -N \log \lambda_0,$$

where λ_0 is the largest eigenvalue of the transfer matrix T, or $\lambda_0 = 2 + e^J$. Thus

$$Z = -N \log \left(2 + e^J\right).$$

A note on how to diagonalize the matrix T. Although writing $\det(T - \lambda) = 0$ results in a qubic equation for λ, one can realize that T is translationary invariant (that is, $T_{\sigma,\sigma'}$ depends only on $|\sigma - \sigma'|$, understood modulo 3). So its eigenvectors can be guessed right away to be $(1,1,1)$, $(1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i})$, and $(1, e^{-\frac{2\pi}{3}i}, e^{-\frac{4\pi}{3}i})$, and the eigenvalues easily follow.

3. Compute its correlation length.

Solution:

$$\xi = \frac{1}{\log\left(\frac{\lambda_0}{\lambda_1}\right)}.$$

Here λ_1 is first eigenvalue of T less than λ_0, or $\lambda_1 = e^J - 1$ (it is actually doubly degenerate eigenvalue). Thus

$$\xi = \frac{1}{\log\left(\frac{e^J+2}{e^J-1}\right)}.$$

Relevant and Irrelevant Operators

Consider a theory in 2 dimensions whose partition function is given by

$$Z = \int \mathcal{D}\phi \; e^{-\int d^2r \; \left[\frac{1}{2\pi}(\nabla\phi)^2 + \lambda_1(\Delta\phi)^2 + \lambda_2\cos(\phi) + \lambda_3\cos(2\phi)\right]}.$$

Are λ_1, λ_2, λ_3 relevant or irrelevant?

Solution: To determine dimension of λ_1 we observe that dimension of ϕ is zero, thus $y_{\lambda_1} + 4 - 2 = 0$ (4 is the dimension of the Laplacian square, while -2 is the dimension of d^2r). Thus

$$y_{\lambda_1} = -2.$$

λ_1 is irrelevant. In general, higher gradients are most always irrelevant and that's why they are almost always thrown out of any theory.

Dimensions of λ_2 and λ_3 are tricker to find because $\cos(\phi)$ has a nontrivial dimension itself. In class we saw that

$$\langle \cos(m\phi(r)) \;\; \cos(m\phi(0)) \rangle = \frac{\int \mathcal{D}\phi \; e^{-\frac{K}{2}\int d^2r(\nabla\phi)^2}\cos(m\phi(r))\cos(m\phi(0))}{\int \mathcal{D}\phi \; e^{-\frac{K}{2}\int d^2r(\nabla\phi)^2}} \sim \left(\frac{a}{r}\right)^{\frac{m^2}{2\pi K}}.$$

It follows from here that the renormalization group dimension of $\cos(m\phi)$ is given by $x_m = \frac{m^2}{4\pi K}$, because the general theory states that

$$\langle \mathcal{O}(r) \; \mathcal{O}(0) \rangle \sim \frac{1}{r^{2x}},$$

where $\mathcal{O}$ is an arbitrary operator (expression involving ϕ) and x is its dimension.

In the problem here $K = 1/\pi$. Thus $x_m = m^2/4$. Since $y = 2 - x$, we find

$$y_{\lambda_2} = 2 - x_1 = 2 - 1/4 = 7/4 > 0,$$

or λ_2 is relevant.

$$y_{\lambda_3} = 2 - x_2 = 2 - 1 = 1 > 0.$$

So λ_3 is also relevant.