

Problem Set 1

Phys 7240
Due: Sep 12

1D Ising Model

The partition function of the 1D Ising model is given by

$$Z = \sum_{\sigma_k = \pm 1} f_L(\sigma_1) \exp \left\{ K \sum_{i=1}^N \sigma_i \sigma_{i+1} + m \sum_{i=1}^{N+1} \sigma_i \right\} f_R(\sigma_{N+1}). \quad (0.1)$$

Here $K = \frac{J}{T}$, and $m = \frac{h}{T}$, where h is the “magnetic field”.

1

There exist a simple trick to compute the partition function of the 1D Ising model in the special case when the external magnetic field is absent, $m = 0$, and when $f_L(\sigma) = f_R(\sigma) = 1$ (free boundary conditions). In other words, when the partition function is

$$Z = \sum_{\sigma_k = \pm 1} \exp \left\{ K \sum_{i=1}^N \sigma_i \sigma_{i+1} \right\}. \quad (1.1)$$

It can be done by making a substitution $\sigma_i \sigma_{i+1} = s_i$ and rewriting the partition function as

$$Z = \sum_{s_k = \pm 1} e^{K \sum_{i=1}^N s_i} = (2 \cosh K)^N. \quad (1.2)$$

Note that this is exact for arbitrary N (no $N \rightarrow \infty$ limit was taken). Note also that this is different from the partition function of the 1D Ising model with periodic boundary conditions, which was shown in class to be $(2 \cosh K)^N + (2 \sinh K)^N$ (Although, of course, it becomes indistinguishable from it in the limit $N \rightarrow \infty$).

1. Explain why the free boundary conditions are important for this trick to work, that is, why Eq. (0.1) with $m = 0$ but with generic functions f_L, f_R , cannot be rewritten as Eq. (1.2), and the substitution for s cannot help to compute the partition function.

2. Compute the partition function Eq. (1.1) by using the transfer matrix technique.

2

Consider a general 1D Ising model, given by Eq. (0.1), with arbitrary m, f_R, f_L .

1. Find its transfer matrix T .
2. Compute the partition function in the limit $N \rightarrow \infty$.
3. Compute the correlation function between the spins, $\langle \sigma_k \sigma_l \rangle - \langle \sigma_k \rangle \langle \sigma_l \rangle$ in the limit of long chain, $N \rightarrow \infty$, and large distance between the spins, $l - k \gg 1$. Find the correlation length ξ .
4. Find the specific heat at $h = 0$, defined as

$$c = -\frac{1}{N} T \frac{\partial^2 F}{\partial T^2},$$

and the spin susceptibility at $h = 0$, defined as

$$\chi = -\frac{1}{N} \left. \frac{\partial^2 F}{\partial h^2} \right|_{h=0}.$$

Compare the answer to the specific heat and the susceptibility of a single spin in a magnetic field, whose partition function is

$$Z = \sum_{\sigma=\pm 1} e^{\frac{h}{T}\sigma}.$$

3

Consider a 1D XY model. It is defined by the variables ϕ_i , each taking values from $-\pi$ to π , which interact with their neighbors ϕ_{i+1} and ϕ_{i-1} so that the partition function is given by

$$Z = \prod_{i=1}^{N+1} \int_{-\pi}^{\pi} d\phi_i \exp \left\{ K \sum_{k=1}^N \cos(\phi_k - \phi_{k+1}) \right\}. \quad (3.1)$$

1. Find the transfer matrix T .
2. Compute the partition function in the limit $N \rightarrow \infty$.
3. Find the correlation length ξ , in the limit $N \rightarrow \infty$.
4. Do the same as in 2-3, but with the help of the substitutions $\varphi_i = \phi_i - \phi_{i+1}$ instead of using the transfer matrix. *Hint:* the correlation function to be computed to find the correlation length is

$$\langle \cos(\phi_n - \phi_l) \rangle = \prod_{i=1}^{N+1} \int_{-\pi}^{\pi} d\phi_i \cos(\phi_n - \phi_l) \exp \left\{ K \sum_{k=1}^N \cos(\phi_k - \phi_{k+1}) \right\}. \quad (3.2)$$