Problem Set 6

Phys 7240

Due: Dec 5

Chain of Josephson Junctions

Figure 1 shows an array of superconducting islands, each having capacitance C_0 to the ground. Each superconducting island is characterized by the quantum mechanical phase ϕ_j. This means the wavefunction of the island is given by $\psi_j = e^{-i\phi_j}$, and the energy of the superconducting island is given by

$$E_j = \frac{i\hbar}{\psi_j}\frac{d\psi_j}{dt} = \hbar\frac{d\phi_j}{dt}.$$

The potential on each island is given by

$$U_j = \frac{\hbar}{2e}\frac{d\phi_j}{dt}.$$

Here e is the electron charge, and in $2e$ we take into account that in superconductivity, the charge propagate in pairs of electrons (Cooper pairs).

When two superconducting islands j and $j+1$ are adjacent to each other (this is called the Josephson junction), the current $I_{j,j+1} = \frac{\hbar}{2eL}\sin(\phi_{j+1}-\phi_j)$ flows from one island to another (this is called the Josephson current, or Josephson effect). Here L can be called the inductance of a junction.

Problem 1. (a) Show that L is indeed similar to the inductance, by expressing $U_{j+1} - U_j$ as $L\dot{I}_{j,j+1}$ ($\dot{I}$ is dI/dt). (b) Calculate the energy of the junction by using that IU is the power of the circuit (integrating in $\int dt\ \left(U_{j+1}-U_j\right) I_{j,j+1}$). (c) Calculate the energy of the islands due to their capacitance to the ground. (d) Calculate the total energy of the chain of N islands. (e) Notice that the terms in the energy due to C_0 can be thought of as kinetic energy (they include time derivative of ϕ), while the terms due to Josephson effect can be thought of as potential energy, show that the action of the chain of islands can be written as

$$S = \sum_j \int dt \left[A\dot{\phi}_j^2 + B\cos(\phi_j - \phi_{j+1}) \right].$$

Calculate A and B.

<u>Problem 2.</u> (a) Using the action above as a starting point, and noting that ϕ plays the role of the coordinate, while $\dot{\phi}$ looks like a momentum, derive quantum mechanical Hamiltonian for the wavefunction $\Psi(\phi_1, \phi_2, \ldots, \phi_N)$ and the Schrödinger equation it satisfies. (b) Alternatively, one can write down a path integral quantization which takes the form

$$Z = \int \prod_j d\phi_j \ e^{\frac{iS}{\hbar}}.$$

Assuming that $|\phi_j - \phi_{j+1}| \ll 1$, derive the continuum version of S. You get the quantum version of XY model. (c) By the appropriate rescaling of x (continuum j) and by choosing $t = -i\tau$ one can bring S to the XY form

$$Z = \int \prod_j d\phi_j \ e^{-\frac{K}{2}\int dx d\tau \ (\nabla\phi)^2}.$$

Find K. (d) Assume that in addition to the capacitance C_0, there is also a mutual capacitance C between the adjacent islands. Derive additional terms which appear in the action S due to C. Is this term relevant, in the RG sense?

<u>Problem 3.</u> Just as the XY model, the chain of Josephson junctions can have vortices. Notice however that ϕ cannot change by a jump in time, only in space. (a) Describe a space-time configuration of ϕ_j which has a vortex. Such a vortex is called a phase-slip in this context. (b) Find the action of interacting vortices, first without the C term (mutual capacitance) and then in the presence of C. (c) Calculate the sine-Gordon action in terms of the dual variables q. Find at what value of L and C_0 the system has the KT transition. (d) Assuming that $\cos q$ term is relevant, expand it in powers of q and show that the resulting system is a chain of islands where the Josephson junctions have been replaced by capacitors. Find the capacitance of these capacitors.

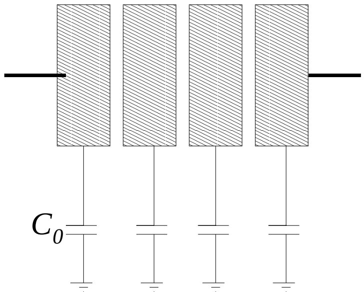

FIG. 1: Superconducting islands, each having capacitance C_0 to the ground.