Problem Set 3

Phys 7240
Due: Oct 10

1 Vortices in a Superconductor

Magnetic field penetrates a superconductor as a thin string - a vortex line. The energy of the vortex line is proportional to its length. If the magnetic field is along the z-direction, then the vortex lines can be described by its $x(z)$ and $y(z)$ coordinates. Therefore, the energy of the vortex line is given by

$$E = \epsilon_0 \int_0^h dz \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} \approx \epsilon_0 \int_0^h dz \, \frac{1}{2} \left[\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right]. \tag{1.1}$$

Here ϵ_0 is the energy per unit vortex line length, and the approximate equality is valid if $\frac{dx}{dz} \ll 1$, $\frac{dy}{dz} \ll 1$.
At a finite temperature the partition function of a vortex is given by

$$Z = \int \mathcal{D}x(z)\mathcal{D}y(z) \; e^{-\frac{E}{T}}, \tag{1.2}$$

where T is the temperature.
Let us assume that the vortex enters the superconductor at a point $x(0) = 0$ and $y(0) = 0$. Calculate the probability that the vortex leaves the superconductor at a point x_f, y_f (so that $x(h) = x_f$, $y(h) = y_f$).

2 Harmonic Oscillator

The action for a harmonic oscillator is given by

$$S = \frac{m}{2} \int_0^T dt \; \left[\dot{x}^2 - \omega_0^2 x^2\right] = \frac{m}{2} \int_0^T dt \; \left\{x \left[-\frac{d^2}{dt^2} - \omega_0^2\right] x\right\}. \tag{2.1}$$

In this problem you will calculate the Green's function of the harmonic oscillator between the points $x(0) = 0$ and $x(T) = 0$, $G(0, 0; T)$, using the approach of operator determinants.

$$G(0, 0; T) \propto \frac{1}{\sqrt{\det\left(-\frac{d^2}{dt^2} - \omega_0^2\right)}}. \tag{2.2}$$

1. Find the eigenvalues λ_n of the operator $-\frac{d^2}{dt^2} - \omega_0^2$ by solving

$$\left[-\frac{d^2}{dt^2} - \omega_0^2\right] \psi_n(t) = \lambda_n \psi_n(t) \tag{2.3}$$

with the boundary conditions $\psi(0) = 0$, $\psi(T) = 0$.

The product $\prod_n \lambda_n$ is divergent. The reason for that is that the factor $\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}$ from the path integral definition is also divergent in the $\epsilon \to 0$, $N \to \infty$ limit in such a way as to compensate the divergence of the determinant. Although it is difficult to take this factor into account, it is possible to avoid dealing with it altogether by introducing $G_0(0, 0; T)$, the Green's function of a free particle from Problem 1, and writing

$$\frac{G(0,0;T)}{G_0(0,0;T)} = \sqrt{\frac{\det\left(-\frac{d^2}{dt^2}\right)}{\det\left(-\frac{d^2}{dt^2} - \omega_0^2\right)}} = \sqrt{\prod_n \frac{\lambda_n^0}{\lambda_n}}, \tag{2.4}$$

λ_n^0 are the eigenvalues of the operator $-\frac{d^2}{dt^2}$, or in other words, they are λ_n if $\omega_0 = 0$. The product in Eq. (2.4) is now convergent.

2. Evaluate the product in Eq. (2.4). Use your knowledge of the Green's function of a free particle from Problem 1 to find $G(0, 0; T)$.

3. Compare the answer with the formula

$$G(x, x_0; t) = \sum_n e^{-iE_n t}\psi_n(x)\psi_n^*(x_0). \tag{2.5}$$

What does it tell you about the energy levels E_n of the harmonic oscillator?

3 Correlation Functions of the Harmonic Oscillator

The action for a harmonic oscillator (in imaginary time) is given by

$$S = \frac{m}{2}\int d\tau \, \left[\dot{x}^2 + \omega_0^2 x^2\right] = \frac{m}{2}\int d\tau \, \left\{x\left[-\frac{d^2}{d\tau^2} + \omega_0^2\right]x\right\}. \tag{3.1}$$

The correlation function $D(\tau)$ for a harmonic oscillator can be defined in two equivalent ways

$$D(\tau) = \langle 0| T\hat{x}(\tau)\hat{x}(0) |0\rangle = \frac{1}{\int \mathcal{D}x \, e^{-S}} \int \mathcal{D}x \, x(0)x(\tau)e^{-S}. \tag{3.2}$$

Here $\hat{x}(\tau)$ is the coordinate operator in the Heisenberg representation $x(\tau) = e^{\hat{H}\tau}xe^{-\hat{H}\tau}$, and T stands for (imaginary) time ordering. The correlation function can easily be calculated if one uses that the integral (even a functional integral) over a total derivative is zero.

$$0 = \int \mathcal{D}x \frac{\delta}{\delta x(\tau)}\left[x(0)e^{-S}\right] = \delta(t)\int \mathcal{D}x \, e^{-S} - m\left[-\frac{d^2}{d\tau^2} + \omega^2\right]\int \mathcal{D}x \, x(\tau)x(0)e^{-S}. \tag{3.3}$$

Consequently,

$$m\left[-\frac{d^2}{d\tau^2} + \omega_0^2\right] D(\tau) = \delta(\tau). \tag{3.4}$$

1. Calculate $D(\tau)$ by solving Eq. (3.4) with the boundary conditions at $\tau \to \pm\infty$ that the correlation function vanishes in this limit.
2. Calculate $D(\tau)$ in an alternative way by using

$$D(\tau) = \langle 0| T\hat{x}(\tau)\hat{x}(0) |0\rangle = \sum_n |\langle n| \hat{x}(0) |0\rangle|^2 e^{-(E_n - E_0)\tau} \tag{3.5}$$

3. Knowing the correlation functions allows one to deduce the commutation relations between the operators. For example,

$$\lim_{\epsilon \to 0} \left[\dot{D}(\epsilon) - \dot{D}(-\epsilon)\right] = \langle 0| \dot{\hat{x}}(0)\hat{x}(0) - \hat{x}(0)\dot{\hat{x}}(0) |0\rangle \,. \tag{3.6}$$

Use $D(\tau)$ you calculated above to figure out the commutation between the operators $\hat{x}$ and $\dot{\hat{x}}$. Using the relation between the momentum $\hat{p} = -i\frac{d}{dx}$ and $\dot{\hat{x}}$, check that you indeed reproduce the commutation relation between the coordinate and the momentum.