

Problem Set 2

Phys 7240
Due: Sep 26

1 Transverse Field Ising Model

The transverse field Ising model is given by the Hamiltonian

$$H = -\gamma \sum_x \tau_x^1 - \beta \sum_x \tau_x^3 \tau_{x+1}^3. \quad (1.1)$$

1. Set $\gamma = 0$ and find in this limit the eigenstates and energy levels of this Hamiltonian.
2. The first excited state of H at $\gamma = 0$ is multiply degenerate. Make γ nonzero but small, and find what happens to the first excited state in perturbation theory in lowest order in γ .

2 Easy Plane Magnet in a Transverse Field

The easy plane magnet in a transverse field is defined as

$$H = -\gamma \sum_x \tau_x^3 - \beta \sum_x \left(\tau_x^1 \tau_{x+1}^1 + \tau_x^2 \tau_{x+1}^2 \right). \quad (2.1)$$

This describes a chain of spins- $\frac{1}{2}$ which prefer to lie in the $x-y$ plane, subject to a magnetic field in the z -direction.

1. Map this into a free fermion problem using the Jordan-Wigner transformation.
2. Find the ground state energy and the correlation length.

3 Bose-Einstein condensation in a weakly interacting gas

A weakly interacting Bose gas at zero temperature is well described by the following Hamiltonian

$$H = \sum_k \left(\frac{k^2}{2m} + 2\Delta \right) a_k^\dagger a_k + \Delta \sum_k \left(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger \right). \quad (3.1)$$

Here a_k, a_k^\dagger are bosonic annihilation and creation operators which satisfy

$$a_k a_k^\dagger - a_k^\dagger a_k = 1, \quad (3.2)$$

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and Δ is related to the density of the Bose-Einstein condensate. For more information about this Hamiltonian, you could see, for example, Landau-Lifshitz, volume 9, chapter on superfluidity.

1. Bring this Hamiltonian to the matrix form

$$H = \sum_k (a_k^\dagger \quad a_{-k}^\dagger \quad a_k \quad a_{-k}) \mathcal{H}_k \begin{pmatrix} a_k \\ a_{-k}^\dagger \\ a_k^\dagger \\ a_{-k} \end{pmatrix} \equiv \sum_k c_k^\dagger \mathcal{H}_k c_k. \quad (3.3)$$

2. Find the conditions a matrix U must satisfy so that $d = Uc$ would still satisfy the bosonic commutation relations Eq. (3.2).

3. Diagonalize the matrix \mathcal{H}_k with the help of the transformation defined by U and find the spectrum of the Hamiltonian Eq. (3.1). This is the spectrum of phonon excitations in a weakly interacting BEC.