1 Transverse Field Ising Model

The transverse field Ising model is given by the Hamiltonian

\[ H = -\gamma \sum_x \tau_x^1 - \beta \sum_x \tau_x^3 \tau_{x+1}^3. \]  

(1.1)

1. Set \( \gamma = 0 \) and find in this limit the eigenstates and energy levels of this Hamiltonian.
2. The first excited state of \( H \) at \( \gamma = 0 \) is multiply degenerate. Make \( \gamma \) nonzero but small, and find what happens to the first excited state in perturbation theory in lowest order in \( \gamma \).

2 Easy Plane Magnet in a Transverse Field

The easy plane magnet in a transverse field is defined as

\[ H = -\gamma \sum_x \tau_x^3 - \beta \sum_x \left( \tau_x^1 \tau_{x+1}^1 + \tau_x^2 \tau_{x+1}^2 \right). \]  

(2.1)

This describes a chain of spins-\( \frac{1}{2} \) which prefer to lie in the \( x-y \) plane, subject to a magnetic field in the \( z \)-direction.
1. Map this into a free fermion problem using the Jordan-Wigner transformation.
2. Find the ground state energy and the correlation length.

3 Bose-Einstein condensation in a weekly interacting gas

A weekly interacting bose gas at zero temperature is well described by the following Hamiltonian

\[ H = \sum_k \left( \frac{k^2}{2m} + 2\Delta \right) a_k^\dagger a_k + \Delta \sum_k \left( a_k a_{-k} + a_k^\dagger a_{-k}^\dagger \right). \]  

(3.1)

Here \( a_k, a_k^\dagger \) are bosonic annihilation and creation operators which satisfy

\[ a_k a_k^\dagger - a_k^\dagger a_k = 1, \]  

(3.2)

continued on the next page
and $\Delta$ is related to the density of the Bose-Einstein condensate. For more information about this Hamiltonian, you could see, for example, Landau-Lifshitz, volume 9, chapter on superfluidity.

1. Bring this Hamiltonian to the matrix form

$$H = \sum_k \begin{pmatrix} a_k & a_{-k} & a_k & a_{-k} \end{pmatrix} \mathcal{H}_k \begin{pmatrix} a_k \\ a_{-k} \\ a_k \\ a_{-k} \end{pmatrix} \equiv \sum_k c_k^\dagger \mathcal{H}_k c_k.$$  \hspace{1cm} (3.3)

2. Find the conditions a matrix $U$ must satisfy so that $d = Uc$ would still satisfy the bosonic commutation relations Eq. (3.2).

3. Diagonalize the matrix $\mathcal{H}_k$ with the help of the transformation defined by $U$ and find the spectrum of the Hamiltonian Eq. (3.1). This is the spectrum of phonon excitations in a weakly interacting BEC.