Phys 7230

Solutions to problem set 6

Problem 1

a. Take some amount of the atmosphere which rises up and expands adiabatically thus cooling down. For an ideal, adiabatically expanding, gas

$$PV^\gamma = P\left(NTP\right)^\gamma = N^\gamma T^\gamma P^{1-\gamma} = \text{const} \tag{1}$$

Here $\gamma = c_P/c_V = 7/5$ for the diatomic ideal gas.

It follows that

$$\frac{dT}{dP} = \frac{2T}{7P}. \tag{2}$$

At the same time, according to the barometric equation,

$$\frac{dP}{dh} = -\frac{mg}{T}P. \tag{3}$$

Expressing dP in terms of dT, we find

$$\frac{dT}{dh} = -\frac{2}{7}mg. \tag{4}$$

With the air molecules weighing, on the average, $4.81 * 10^{-26}$kg, we find

$$\frac{dT}{dh} \approx -9.7K/km. \tag{5}$$

b. Now suppose we have some number of molecules of water vapor in the air, N_v. As air rises, not only its pressure drops by dP, but also some number of molecules of vapor $-dN_0$ condense. For definiteness, dN_v is the increase of the number of molecules of vapor, so that dN_v is accompanies by the consumption of energy LdN_v. As the energy is consumed, the temperature drops.

Write down the energy conservation relation (first law of thermodynamics)

$$\frac{5}{2}NdT = -LdN_v - PdV. \tag{6}$$

Use the equation of state

$$PV = NT \ \rightarrow\ dV = \frac{NdT}{P} - \frac{NT}{P^2}dP \tag{7}$$

to find

$$\frac{5}{2}NdT = -LdN_v - NdT + \frac{NT}{P}dP \ \rightarrow\ dT = -\frac{2}{7}\frac{L}{N}dN_v + \frac{2}{7}\frac{T}{P}dP. \tag{8}$$

c.

Both vapor and air can be thought of as ideal gases occupying the same volume V at the same temperature T. Thus they obey

$$\frac{N_v}{P_v} = \frac{N}{P}. \tag{9}$$

This gives

$$dN_v = \frac{NdP_v}{P} - \frac{NP_v}{P^2}dP. \tag{10}$$

At the same time, the Clapeyron-Clausius equation states

$$\frac{dP_v}{dT} = \frac{LP_v}{T^2}. \tag{11}$$

This leads to

$$dN_v = \frac{NLP_v}{PT^2} dT - \frac{NP_v}{P^2} dP. \tag{12}$$

Substituting into the equation derived in part **b** we find

$$dT \left(1 + \frac{2}{7} \frac{L^2 P_v}{PT^2} \right) = \frac{2}{7} \frac{T}{P} \left(1 + \frac{LP_v}{PT} \right) dP. \tag{13}$$

This generalizes (2).

Now we express dP in terms of dT and substitute into the barometric equation. This gives

$$\frac{dT}{dh} = -\frac{2}{7} mg \frac{1 + \frac{L}{T} \frac{P_v}{P}}{1 + \frac{2}{7} \frac{L^2}{T^2} \frac{P_v}{P}}. \tag{14}$$

Now numbers. $L \approx 2.3 \cdot 10^6 J/kg$, which is equivalent to $7 \cdot 10^{-20} J/$water molecule, or $5000K/molecule$. $P_v/P \approx 0.03$ at $25°C$. Then

$$\frac{1 + \frac{L}{T} \frac{P_v}{P}}{1 + \frac{2}{7} \frac{L^2}{T^2} \frac{P_v}{P}} \approx 0.45 \tag{15}$$

and so the lapse rate is about $-4.5K/km$. (I took the numbers from various websites).

Problem 2

In the one-dimensional Ising model the partition function is

$$Z = \sum_{\sigma_i = \pm 1} e^{\frac{J}{T} \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1}} \tag{16}$$

We wish to calculate the correlation function

$$\langle \sigma_m \sigma_n \rangle = \frac{1}{Z} \sum_{\sigma_i = \pm 1} \sigma_m \sigma_n e^{\frac{J}{T} \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1}} \tag{17}$$

First do the transformation of variables

$$s_k = \sigma_k \sigma_{k+1} \tag{18}$$

Then the partition function becomes

$$\begin{aligned} Z &= 2 \sum_{s_i = \pm 1} \prod_{k=1}^{N-1} e^{\frac{J}{T} s_k} \\ &= 2 \left(\sum_{s_1 = \pm 1} e^{\frac{J}{T} s_1} \right) \left(\sum_{s_2 = \pm 1} e^{\frac{J}{T} s_2} \right) \cdots \\ &= 2 \prod_{k=1}^{N-1} \sum_{s_k = \pm 1} e^{\frac{J}{T} s_k} \end{aligned} \tag{19}$$

The factor two arises from the fact that the set of variables $\{s_k\}$ is invariant under a reversal of all the spins in the chain.

Since, for $m < n$,

$$\sigma_m \sigma_n = \prod_{k=m}^{n-1} s_k \tag{20}$$

the correlation function is

$$\begin{aligned}
\langle \sigma_m \sigma_n \rangle &= \frac{2}{Z} \sum_{s_i=\pm 1} \prod_{j=m}^{n-1} s_j \prod_{k=1}^{N-1} e^{\frac{J}{T} s_k} \\
&= \frac{2}{Z} \sum_{s_1=\pm 1} e^{\frac{J}{T} s_1} \cdots \sum_{s_{m-1}=\pm 1} e^{\frac{J}{T} s_{m-1}} \\
&\quad \times \sum_{s_m=\pm 1} s_m e^{\frac{J}{T} s_m} \cdots \sum_{s_{n-1}=\pm 1} s_{n-1} e^{\frac{J}{T} s_{n-1}} \sum_{s_n=\pm 1} e^{\frac{J}{T} s_n} \cdots \sum_{s_{N-1}=\pm 1} e^{\frac{J}{T} s_{N-1}}
\end{aligned} \tag{21}$$

It is seen that there is great cancellation between the terms in the numerator and the denominator in the correlation function. After cancelling common factors, we find

$$\langle \sigma_m \sigma_n \rangle = \prod_{k=m}^{n-1} \frac{\sum_{s_k=\pm 1} s_k e^{\frac{J}{T} s_k}}{\sum_{s_k=\pm 1} e^{\frac{J}{T} s_k}} = \prod_{k=m}^{n-1} \tanh \frac{J}{T} \tag{22}$$

Thus it is seen that

$$\langle \sigma_m \sigma_n \rangle = \left[\tanh \left(\frac{J}{T} \right) \right]^{n-m} = e^{(n-m) \ln \tanh(J/T)} \tag{23}$$

Here we assumed that $m < n$. It is clear that the generalization of the above result is

$$\langle \sigma_m \sigma_n \rangle = e^{-|m-n| \ln \coth(J/T)} \tag{24}$$

The correlation length is thus

$$l = \frac{1}{\ln \coth(J/T)} \tag{25}$$

As temperature goes to zero, $\coth(J/T)$ approaches 1, and the correlation length diverges.

Problem 3

The free energy of a ferromagnet is

$$\frac{F}{T} = \int d^3x \left[\kappa \left(\nabla M \right)^2 + \alpha (T - T_c) M^2 + \lambda M^4 - hM \right] \tag{26}$$

In this problem we set $T = T_c$, the critical temperature beneath which the system behaves ferromagnetic, and compute the correlation function $\langle M(x) M(y) \rangle$.

First we find the magnetization M_0 which minimizes the free energy. Assuming that this is a constant it is determined by

$$4\lambda M_0^3 - h = 0 \tag{27}$$

with solution

$$M_0 = \left(\frac{h}{4\lambda} \right)^{1/3} \tag{28}$$

Now let

$$M(x) = M_0 + \eta(x) \tag{29}$$

Assuming that fluctuations around the minimum are small we expand the free energy to second order in these to find

$$\frac{F}{T} \approx \frac{F_0}{T} + \int d^3x \left[\kappa \left(\nabla \eta \right)^2 + 4\lambda M_0^3 \eta + 6\lambda M_0^2 \eta^2 - h\eta \right] \tag{30}$$

where F_0 is the value of the free energy at the minimum, M_0. F_0 will not influence the correlation function since it cancels in taking the expectation value. Also, the linear terms in η do not cause fluctuations and can be ignored in the following.

Fourier transform the fluctuations

$$\eta(x) = \sum_p e^{-i\vec{p}\cdot\vec{x}} \eta_{\vec{p}} \tag{31}$$

Then the relevant part of the free energy is

$$\frac{F'}{T} = \sum_p \left[\kappa p^2 + 6\lambda M_0^2\right] \eta_p \eta_{-p} \tag{32}$$

Thus

$$\langle \eta_p \eta_{-p} \rangle = \frac{1}{2(\kappa p^2 + 6\lambda M_0^2)} \tag{33}$$

and

$$\begin{aligned} \langle \eta(x)\eta(y) \rangle &= \frac{1}{V}\sum_p \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2(\kappa p^2 + 6\lambda M_0^2)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2(\kappa p^2 + 6\lambda M_0^2)} \\ &= \frac{e^{-r/l}}{8\pi\kappa r} \end{aligned} \tag{34}$$

where $r \equiv |\vec{x} - \vec{y}|$ and $l = (\kappa/(6\lambda M_0^2))^{1/2}$.

Problem 4

We study here the Gibbs potential

$$G(M) = aM^2 + bM^4 + cM^6, \qquad c > 0 \tag{35}$$

Note that the Gibbs potential will have extrema at the values

$$M_0 = 0 \quad \text{or} \quad M_0^2 = -\frac{b}{3c} \pm \frac{\sqrt{b^2 - 3ac}}{3c} \tag{36}$$

Here the extrema should be chosen such that M_0^2 is positive. Note that the coefficients a and b depend on both the pressure and the temperature.

If $a > 0$ and $b > 0$ the potential has a minimum at zero magnetization. If b is kept positive while a goes through zero and becomes negative the minimum will be at $M_0^2 > 0$ and the system will develop into this new minimum. The magnetization is continuous across the phase transition and thus this is a second order phase transition. The coefficients a and b can be approximated close to the line of second order phase transitions to be

$$\begin{aligned} a(P,T) &= \alpha(P)(T - T_c) \\ b(P,T) &= \beta(T)(P - P_{tc})) \end{aligned} \tag{37}$$

P_{tc} is the value of the pressure at the tricritical point where b vanishes. α and β are slowly varying functions. Using these expansions we can consider the change in entropy and specific heat. Since the Gibbs potential vanishes for temperatures above the transition temperature these changes reduce to

$$\begin{aligned} \Delta S &= \left.\frac{\partial G}{\partial T}\right|_{T=T_c^-} \\ &= 0 \end{aligned} \tag{38}$$

and

$$\Delta C = T \left. \frac{\partial^2 G}{\partial T^2} \right|_{T=T_c^-} = -\frac{\alpha^2 T_c}{2b} \tag{39}$$

Now assume $b < 0$. For $a < 0$ the system has two minima located at $|M| > 0$ and no phase transition occurs. In the range $0 < a < b^2/3c$ there are five extrema. This is illustrated in the

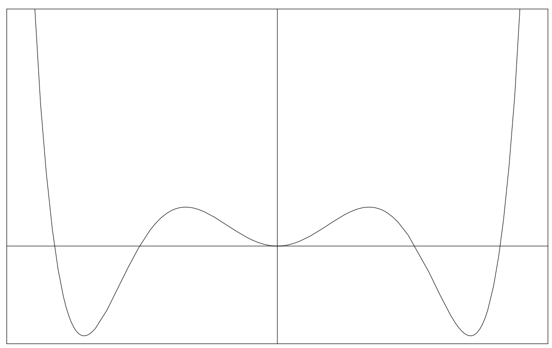

figure. There will now be three local minima and the phase transition occurs when the value of the Gibbs potential at 0 coincides with the value of the potential at the other minima. At this point the magnetization will have a discontinuity. Solving for the minimum at $M > 0$ we define $x \equiv M^2$ and find the minimum to be located at

$$x_{min} = -\frac{b}{2c} \tag{40}$$

The value of the potential here is

$$a - \frac{b^2}{4c} \tag{41}$$

and thus the first order phase transition occurs at the line $a = \frac{b^2}{4c}$ (for $b < 0$). It is seen that this is within the range $0 < a < b^2/3c$ as it should be.

Now we expand

$$\begin{aligned} a(P,T) &= \frac{b^2}{4c} + \gamma(P)(T - T_c) \\ b(P,T) &= \delta(T)(P - P_{tc}) \end{aligned} \tag{42}$$

and find

$$\begin{aligned} \Delta S &= -\frac{\gamma b}{6c} \\ \Delta C &= -\frac{T_c \gamma^2}{b} \end{aligned} \tag{43}$$

Note that since $b < 0$ the change in entropy is positive as temperature is increasing.