

Phys 7230

Problem Set 6

Due: Apr 27

1

In this (long, instructive, but not particularly difficult) problem, since we are in Boulder and NCAR is located nearby, we will study the temperature of the Earth's atmosphere. According to the basic principles of statistical physics and thermodynamics, the atmosphere in thermodynamic equilibrium would have uniform temperature T . At the same time, the probability to find a molecule at an altitude h goes as $\exp(-mgh/T)$ (Boltzmann distribution, g is the free fall acceleration, h is the altitude, m is the mass of the molecule), which implies the atmospheric density should go as $\rho = N/V = \rho_0 \exp(-mgh/T)$, where ρ_0 is the density at $h = 0$. Since $P = \rho T$ (ideal gas equation of state), it follows that the pressure also obeys $P = P_0 \exp(-mgh/T)$. Another way to derive this involves considering a horizontal slab of air, with the pressure from the air below pushing it up, while the pressure from the air above and the gravity pushing it down. The condition for the mechanical equilibrium plus the equation of state of the ideal gas immediately leads to $dP/dh = -mgP/T$ (called the barometric equation). Now while this formula does sometimes work, it cannot be the whole story. Indeed, the temperature in practice drops quickly with the altitude, with the temperature outside an airliner at a cruise altitude being close to $-60C$. The resolution is simple: the Sun heats the surface of the Earth, while the heat propagates through the air very slowly (air is a good heat insulator, think of the double-pane windows). As a result, the air close to the surface of the Earth is hotter than the air higher up. However, a question arises: why wouldn't the hotter air next to the surface rise up to higher altitudes (like the air inside the hot air balloon) and mix up different atmospheric layers thus equilibrating the temperature (the process called convection)? The reason this does not happen is the following: as the air rises up, it expands adiabatically (remember that the heat exchange between different patches of the air is very slow), thus cooling down. The condition for the convection to occur is that the raising air mass must remain warmer than the surrounding air despite this adiabatic cooling.

a. Assuming that dT/dh is just at the critical value for convection to begin, compute this quantity. The result should be a constant, independent of the temperature, which evaluates roughly to $-10^\circ C/km$. This fundamental meteorological quantity is known as the dry adiabatic lapse rate. Use that the specific heat for the air is given by $c_V = 5/2$ (air consists mostly of the diatomic molecules O_2 and N_2). Since the air is 1/3 oxygen and 2/3 nitrogen, replace m with the weighted average of their respective masses.

This is not yet the whole story either. Often the air is saturated with water vapor. As the raising mass of air experiences a drop in pressure, the vapor condenses. This releases heat, which leads to the air being hotter than in the dry case. Thus the convection may start at a different temperature gradient. The goal is to calculate this critical gradient. Assume that the saturated vapor takes up a small fraction of the atmospheric density in all calculations. This problem is not straightforward, so let us split it into two steps:

b. Show that, as the air undergoes adiabatic expansion while the vapor condenses, the change in temperature dT can be related to the change in pressure dP and the number of water vapor molecules which undergo condensation $-dN_v$ according to

$$dT = \frac{2}{7} \frac{T dP}{P} - \frac{2}{7} \frac{L}{N} dN_v, \quad (1.1)$$

where L is the latent heat of vaporization per molecule, N_v is the number of molecules of vapor (so that N_v decreases during the condensation process and dN_v is negative) and N is the total number of molecules in the air (such that $N_v \ll N$). This equation replaces the equation for the adiabatic expansion of the dry gas.

c. As the air expands and cools, the water vapor remains saturated (by continuously condensing). By relating N_v to N , P , and P_v (the vapor pressure), find dN_v in terms of dT and dP , the vapor pressure P_v , and dP_v . Use the Clapeyron-Clausius equation to find dP_v/dT . Combine it with the equation found in part **b** to compute dP/dT under these conditions. Next, combine what you found with the barometric formula to find the new dT/dh , called the wet adiabatic lapse rate. It should be a function of m , P_v , P , T , and L . Finally, calculate this rate at atmospheric pressure and room temperature (25°C).

2

An Ising model was introduced by Lenz. It is a model of a magnet. The partition function of the one dimensional Ising model is defined in the following way

$$Z = \sum_{\sigma_i = \pm 1} e^{\frac{J}{T} \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1}}.$$

Here σ_k are variables, taking values of $+1$ and -1 corresponding to spins pointing up or down, J is some parameter having dimensions of energy which measures how strongly the spins interact, and T is temperature.

Calculate the correlation function between the spins σ , defined as

$$\langle \sigma_m \sigma_n \rangle = \frac{1}{Z} \sum_{\sigma_i = \pm 1} \sigma_m \sigma_n e^{\frac{J}{T} \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1}}$$

and find the correlation length l as a function of the temperature T (and J , of course). The correlation length is defined by

$$\langle \sigma_m \sigma_n \rangle \sim e^{-\frac{|m-n|}{l}}, \quad |m-n| \gg l.$$

What does the correlation length do as temperature is taken to zero?

You may use any method you like, although the following trick could be useful: define new variables s_k such that

$$s_k = \sigma_k \sigma_{k+1}.$$

Notice that (assuming that $m < n$)

$$\sigma_m \sigma_n = \prod_{k=m}^{n-1} s_k.$$

Replace summation over $\sigma_k = \pm 1$ by $s_k = \pm 1$.

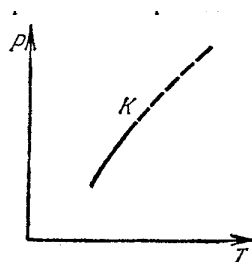
3

The free energy of a ferromagnet with magnetization $M(x)$ can be written as

$$\frac{F}{T} = \int d^3x \left[\kappa (\vec{\nabla} M)^2 + \alpha(T - T_c)M^2 + \lambda M^4 - hM \right].$$

Here T_c is the temperature below which the system behaves as a ferromagnet, and h is proportional to the external magnetic field. Take $T = T_c$ and find the correlation function $\langle M(x)M(y) \rangle$ at $h \neq 0$. *Hint:* Expand F in powers of M around its minimum up to quadratic terms.

4



As pressure is varied, the critical temperature of a second order phase transition can change as well. At a certain pressure, the second order phase transitions can get replaced by the first order phase transitions. The figure above illustrates the the curve of the critical temperature of the second order phase transitions (dotted line), and the point K where the second order phase transitions get replaced by first order phase transitions, shown by the solid line. The point K is called the tricritical point, for historic reasons.

This situation occurs in a magnet with vacancies, for example (see the book by J. Cardy, *Scaling and renormalization in statistical physics*, or many other books, for relevant discussions).

To describe this situation, an expansion of the Gibbs potential is introduced up to the terms of the sixth order

$$G(M) = a(T, P) M^2 + b(T, P) M^4 + c M^6.$$

Here $c > 0$, while $a(T, P)$ and $b(T, P)$ can be of either sign (c also depends on T and P , however as it stays positive, its dependence on temperature and pressure does not alter physics and can be neglected). At the tricritical point $a = 0$, and $b = 0$.

Find the values of a and b which correspond to the line of second order transitions and first order transitions. Find the behavior of the Gibbs potential and the specific heat c_P as the line of the transitions is crossed.