(1) Shankar 17.2.2

$$\hat{H} = \hat{H}_0 + \hat{H}' \qquad \hat{H}_0 = -\gamma B_0 \hat{S}_z \quad \hat{H}' = -\gamma B \hat{S}_x$$

Unperturbed states are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in $\hat{S}_z$ basis,

energies $\frac{-\hbar\gamma B_0}{2}$ and $+\frac{\hbar\gamma B_0}{2}$. $\begin{pmatrix}1\\0\end{pmatrix} \leftrightarrow |+_z\rangle$ ground state.

call states $|0^{(0)}\rangle$ and $|1^{(0)}\rangle$.

$$E_0^{(1)} = \langle 0^{(0)}|\hat{H}'|0^{(0)}\rangle = -\gamma B\langle +_z|\hat{S}_x|+_z\rangle = 0$$

$$E_0^{(2)} = \sum_{i\neq 0}^{\infty} |\langle i^{(0)}|\hat{H}'|0^{(0)}\rangle|^2 / (E_0^{(0)} - E_i^{(0)})$$

$$= (\gamma B)^2 \frac{|\langle -_z|S_x|+_z\rangle|^2}{-\hbar\gamma B_0} = -\frac{\hbar\gamma}{4}\frac{B^2}{B_0}$$

$$\langle 1^{(0)}|0^{(1)}\rangle = \frac{\langle 1^{(0)}|\hat{H}'|0^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}} = \frac{-\hbar\gamma B}{2}\frac{1}{-\hbar\gamma B_0} = \frac{B}{2B_0}$$

$$\text{so } E_0 = \frac{-\hbar\gamma B_0}{2} - \frac{\hbar\gamma}{4}\frac{B^2}{B_0} + \cdots$$

$$|0\rangle = |+_z\rangle + \frac{B}{2B_0}|-_z\rangle + \cdots$$

Exact: Recall that we can rotate a spinor into a new basis by:

$$|\vec{e}_{\vec{n}}\cdot\vec{S}:+\rangle = \begin{pmatrix}\cos\frac{\beta}{2}e^{-i\alpha/2}\\ \sin\frac{\beta}{2}e^{i\alpha/2}\end{pmatrix} \quad \text{where } \vec{n} = B_0\vec{e}_z + B\vec{e}_x \Rightarrow \tan\beta = \frac{B}{B_0}$$

$$\vec{e}_n = \frac{\vec{n}}{|\vec{n}|} = \frac{B_0\vec{e}_x + B\vec{e}_z}{\sqrt{B_0^2+B^2}}$$

So $E_0 = \frac{-\hbar\gamma}{2}\sqrt{B_0^2+B^2} = \frac{-\hbar\gamma}{2}B_0\left[1+\frac{B^2}{2B_0^2}+\mathcal{O}(B^4)\right]$

$= \frac{-\hbar\gamma B_0}{2} - \frac{\hbar\gamma}{2}\frac{B^2}{B_0} + \mathcal{O}(B^4)$ ✓

and $|0\rangle = \cos\frac{\beta}{2}\begin{pmatrix}1\\ \tan\frac{\beta}{2}\end{pmatrix}$

$= \underbrace{\cos\frac{\beta}{2}}_{=1 \text{ to order } \beta}\left[\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\left(\frac{B}{2B_0}\right)\right] = |0^{(0)}\rangle + \frac{B}{2B_0}|1^{(0)}\rangle + \dots$ ✓

(2) Shankar 17.2.3

proton has charge $+e$.

1) By Gauss' law, $\vec{E} = \frac{+e}{r^2}\vec{e}_r$ for $r>R$

$$= \frac{+e}{r^2}\left(\frac{r^3}{R^3}\right) = \frac{+er}{R^3} \quad \text{for } r<R$$

so $\Phi(r) = \int_r^\infty dr' E(r') \quad = \frac{+e}{r}$ for $r>R$

$$= \frac{+e}{r} + \int_r^R dr'\left(+\frac{er'}{R^3}\right) = +\frac{3}{2}\frac{e}{R} - \frac{er^2}{2R^3} \quad \text{for } r<R$$

Potential energy $V(r) = -e\Phi(r)$

$$= \begin{cases} -\frac{e^2}{r} & r>R \\ -\frac{3}{2}\frac{e^2}{R} + \frac{e^2 r^2}{2R^3} & r<R \end{cases}$$

2) $\hat{H} = \hat{H}_0 + \hat{H}'$ Let $\hat{H}_0 = \frac{-e^2}{r} \equiv V_0(r)$

Then $\hat{H}' = V(r) - V_0(r)$

$$= 0 \quad r>R$$

$$= \frac{-3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r} \quad r<R$$

Energy shift $E_0^{(1)} = \langle 100|\hat{H}'|100\rangle$

$$= \int_0^R dr\, r^2\, \frac{4e^{-2r/a_0}}{a_0^3}\left(-\frac{3}{2}\frac{e^2}{R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r}\right)$$

Now take $e^{-2r/a_0} \approx 1$.

$$E_0^{(1)} \approx \frac{4}{a_0^3}\int_0^R dr \left(-\frac{3}{2}\frac{e^2 r^2}{R} + \frac{e^2 r^4}{2R^3} + e^2 r \right)$$

$$= \frac{4}{a_0^3}\left(\frac{-3e^2}{2R}\frac{R^3}{3} + \frac{e^2}{2R^3}\frac{R^5}{5} + e^2\frac{R^2}{2} \right)$$

$$= \frac{2e^2R^2}{5a_0^3}.$$

(3) Shankar 17.3.2

$$\hat{H}_0 = A\hat{S}_z^2 \qquad \hat{H}' = B(\hat{S}_x^2 - \hat{S}_y^2)$$

$\hat{H}_0$ has 2 eigenvalues: 0 (nondegenerate: $|0\rangle$ in S_z basis)
$A\hbar^2$ (2-fold degenerate: $|+\rangle$ and $|-\rangle$ in S_z basis)

Consider the degenerate subspace in the S_z basis: $|+\rangle, |-\rangle$.
Diagonalize $\hat{H}'$ in this subspace:

$$\hat{H}' \Leftrightarrow \begin{pmatrix} H'_{++} & H'_{+-} \\ H'_{-+} & H'_{--} \end{pmatrix} \quad \text{where } H'_{+-} = \langle + | \hat{H}' | - \rangle \text{ etc.}$$

In $\hat{S}_z$ basis, $$\hat{H}' \leftrightarrow B\hbar^2 \left[\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^2 - \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}^2 \right]$$

$$= B\hbar^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so $\hat{H}' \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B\hbar^2$ in sub-basis. This maximally mixes the $S_z = \pm\hbar$ states; its eigenvectors are $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, which are the stable states under the (1st order) perturbation.

Exact states: in S_z basis, full Hamiltonian is just

$$\hat{H} \leftrightarrow \hbar^2\left[\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix} + \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B & 0 & 0 \end{pmatrix}\right] = \hbar^2\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Which has the same eigenvectors as indicated by perturbation calculation: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

(4) Shankar 17.3.3

Spin-orbit correction is $\frac{1}{4}mc^2\alpha^4 \frac{j(j+1)-l(l+1)-\frac{3}{4}}{n^3 l(l+\frac{1}{2})(l+1)}$

$$= \frac{1}{4}mc^2\alpha^4 \frac{l\delta_{j,l+\frac{1}{2}} - (l+1)\delta_{j,l-\frac{1}{2}}}{n^3 l(l+\frac{1}{2})(l+1)}$$

Now consider perturbation $\hat{H} = -\hat{\mu}_z B$

$$\hat{\mu}_z = \frac{e}{2mc}\hat{L}_z + \frac{ge}{2mc}\hat{S}_z \quad \text{Take } g\approx 2.$$

So $\hat{\mu}_z = \frac{e}{2mc}(\hat{L}_z + 2\hat{S}_z)$

$$= \frac{e}{2mc}(\hat{J}_z + \hat{S}_z) \quad \text{in coupled}$$

$$\hat{H}' = -\frac{eB}{2mc}(\hat{J}_z + \hat{S}_z) \quad \text{but of course } \hat{J}_z, \hat{S}_z \text{ don't commute.}$$

First order shift = expectation value

$$\frac{-eB}{2mc}\langle J_z + S_z\rangle_{j=l\pm\frac{1}{2},\, m_j}$$

$$= \frac{-eB}{2mc}\left[m_j\hbar + \langle S_z\rangle\right] = \frac{-eB}{2mc}\left[j_z + \langle S_z\rangle\right]$$

Need $|j=l\pm\frac{1}{2}, m\rangle$ in S_z basis:

$$|j=l\pm\tfrac{1}{2}, m\rangle = C_+|m_l = m_j - \tfrac{1}{2}, m_s = \tfrac{1}{2}\rangle + C_-|m_l = m_j + \tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle$$

$C_\pm$ Clebsch-Gordan coeffs are $C_+ = \pm\sqrt{\frac{l\pm m+\frac{1}{2}}{2l+1}}$ $\quad C_- = \sqrt{\frac{l\mp m+\frac{1}{2}}{2l+1}}$

$$\langle S_z\rangle = \frac{\hbar}{2}|C_+|^2 - \frac{\hbar}{2}|C_-|^2 = \frac{\hbar}{2(2l+1)}\left(l\pm m+\tfrac{1}{2} - l\pm m - \tfrac{1}{2}\right) = \frac{\pm m\hbar}{2l+1} = \frac{\pm j_z}{2l+1}$$

$$\Rightarrow E \text{ shift} = \frac{-eB}{2mc}\left[1 \pm \frac{1}{2l+1}\right]j_z \quad \text{QED.}$$

(5)

$$\hat{H}_0 = \frac{\hat{P}_x^2}{2m} + \frac{\hat{P}_y^2}{2m} + \frac{1}{2} m\omega_0 \left(\hat{x}^2 + \hat{y}^2\right)$$

Define quantum numbers n_x, n_y and operators $\hat{a}_x, \hat{a}_x^+, \hat{a}_y, \hat{a}_y^+$ as usual.

The eigenvalue $3\hbar\omega_0$ is 3 fold degenerate.
In the $\hat{H}_0$ basis, degenerate subspace is spanned by states $|n_x=1, n_y=1\rangle, |n_x=0, n_y=2\rangle, |n_x=2, n_y=0\rangle \equiv |11\rangle, |02\rangle |20\rangle$

$$\hat{H}' = 2\lambda \hat{x}\hat{y} = \frac{\hbar\lambda}{m\omega_0}(\hat{a}_x + \hat{a}_x^+)(\hat{a}_y + \hat{a}_y^+)$$

$$= \frac{\hbar\lambda}{m\omega_0}\left(\hat{a}_x\hat{a}_y + \hat{a}_x^+\hat{a}_y + \hat{a}_y^+\hat{a}_x + \hat{a}_x^+\hat{a}_y^+\right)$$

Now let a generic vector $|\psi\rangle = \psi_1|11\rangle + \psi_2|02\rangle + \psi_3|20\rangle$ in the degenerate subspace. Need to find $|\psi\rangle$ that are eigenvectors of $\hat{H}'$ (in the subspace).

$$\hat{H}'|\psi\rangle = \psi_1 \hat{H}'|11\rangle + \psi_2 \hat{H}'|02\rangle + \psi_3 \hat{H}'|20\rangle.$$

$$= \frac{\hbar\lambda}{m\omega_0}\Big\{\psi_1\left[|00\rangle + \sqrt{2}|02\rangle + \sqrt{2}|20\rangle + 2|22\rangle\right] + \psi_2\left[0 + \sqrt{2}|11\rangle + \sqrt{3}|13\rangle\right]$$

$$+ \psi_3\left[0 + \sqrt{2}|11\rangle + \sqrt{3}|31\rangle\right]$$

Ignore terms outside subspace: $\hat{H}'_{sub} \leftrightarrow \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \frac{\hbar\lambda}{m\omega_0}$

which has eigenvalues $\left[-2, 0, +2\right] \frac{\hbar\lambda}{m\omega_0}$.

5 cont'd

Ground state shift: $\langle 00 | \hat{H}' | 00 \rangle$

$$= \frac{\hbar\lambda}{m\omega_0} \langle 00 | a_x^\dagger a_y^\dagger | 00 \rangle$$ (other three terms annihilate $|00\rangle$)

$$= \frac{\hbar\lambda}{m\omega_0} \langle 00 | 11 \rangle = 0$$

So ground state is unchanged to first order.

6 Exact $\hat{H}$:

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2\right) + 2\lambda\hat{x}\hat{y}$$

$$= \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2\left[\left(\frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{2} + \hat{x}\hat{y}\right) + \left(\frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{2} - \hat{x}\hat{y}\right) + \frac{4\lambda\hat{x}\hat{y}}{m\omega^2}\right]$$

$$= \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2\left[\left(\frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{2} + \hat{x}\hat{y} + \frac{\lambda}{m\omega^2}\hat{x}^2 + \frac{\lambda}{m\omega^2}\hat{y}^2 + \frac{2\lambda\hat{x}\hat{y}}{m\omega^2}\right) + \left(\frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{2} - \hat{x}\hat{y} - \frac{\lambda}{m\omega^2}\hat{x}^2 - \frac{\lambda}{m\omega^2}\hat{y}^2 + \frac{2\lambda\hat{x}\hat{y}}{m\omega^2}\right)\right]$$

$$= \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2\left[\left(1 + \frac{\lambda}{m\omega^2}\right)\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)^2 + \left(1 - \frac{\lambda}{m\omega^2}\right)\left(\frac{\hat{x}-\hat{y}}{\sqrt{2}}\right)^2\right]$$

Let $u \equiv \frac{x+y}{\sqrt{2}}$ $\quad v = \frac{x-y}{\sqrt{2}}$

$$= \frac{1}{2m}\left(\hat{p}_u^2 + \hat{p}_v^2\right) + \frac{1}{2}m\omega^2\left[\left(1 + \frac{\lambda}{m\omega^2}\right)\hat{u}^2 + \left(1 - \frac{\lambda}{m\omega^2}\right)\hat{v}^2\right]$$

$\Rightarrow$ Define $\omega_u^2 = \omega^2\left(1 + \frac{\lambda}{m\omega^2}\right)$ $\quad \omega_v^2 = \omega^2\left(1 - \frac{\lambda}{m\omega^2}\right)$

To first order in λ, $\omega_u = \omega + \frac{\lambda}{2m\omega}$; $\omega_v = \omega - \frac{\lambda}{2m\omega}$

So the eigenvalues of $\hat{H}$ are $\left[\omega_u\left(n_u+\frac{1}{2}\right)+\omega_v\left(n_v+\frac{1}{2}\right)\right]\hbar$

$$= \hbar\left[\left(n_u+n_v+1\right)\omega + \frac{\lambda}{2m\omega}\left(n_u-n_v\right)\right]$$

So the shift for the ground state $(n_u = n_v = 0)$ is zero,

for the 1st excited states:

$n_u = n_v = 1$: zero

$n_u = 2, n_v = 0$: $\frac{\lambda\hbar}{m\omega}$

$n_u = 0, n_v = 2$: $-\frac{\lambda\hbar}{m\omega}$

As expected from pert. theory calculation.