(1) [12.4.2]

Using simpler notation, we're asked to show:

$$\hat{R}_y(-\varepsilon_y)\hat{R}_x(-\varepsilon_x)\hat{R}_y(\varepsilon_y)\hat{R}_x(\varepsilon_x) = \hat{R}_z(-\varepsilon_x\varepsilon_y)$$

$$\hat{R}_x(\varepsilon_x) \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2) \quad ; \quad \hat{R}_y(\varepsilon_y) \Leftrightarrow \begin{pmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z(\varepsilon_z) \Leftrightarrow \begin{pmatrix} 1 & -\varepsilon_z & 0 \\ \varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_y(\varepsilon_y)\hat{R}_x(\varepsilon_x) \overset{1^{st}\text{ order}}{\Leftrightarrow} \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & \varepsilon_y \\ 0 & 1 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 1 \end{pmatrix} \equiv \hat{B}$$

$$\hat{R}_y(-\varepsilon_y)\hat{R}_x(-\varepsilon_x) \Leftrightarrow \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & -\varepsilon_y \\ 0 & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{pmatrix} \equiv \hat{A}$$

$$\hat{A}\hat{B} = \begin{pmatrix} (1+\varepsilon_y^2) & \varepsilon_x\varepsilon_y & -\varepsilon_x^2\varepsilon_y \\ -\varepsilon_x\varepsilon_y & (1+\varepsilon_x^2) & 0 \\ 0 & \varepsilon_x\varepsilon_y^2 & 1+\varepsilon_x^2+\varepsilon_y^2 \end{pmatrix}$$

Zero terms of order $\varepsilon_x^2$, $\varepsilon_y^2$:
$$= \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & 0 \\ -\varepsilon_x\varepsilon_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

QED

② [12.5.3]

1) $\langle \hat{J}_x \rangle = \frac{1}{2}\langle \hat{J}_+ + \hat{J}_- \rangle = \frac{1}{2}\left[\langle \hat{J}_+ \rangle + \langle \hat{J}_- \rangle\right]$

$= \frac{1}{2}\left[\langle jm|\hat{J}_+|jm\rangle + \langle jm|\hat{J}_-|jm\rangle\right]$

$= \frac{1}{2}\left[C^+_{jm}\underbrace{\cancel{\langle jm|j\,m+1\rangle}}_{0} + C^-_{jm}\underbrace{\cancel{\langle jm|j\,m-1\rangle}}_{0}\right]$ (see Eq. 12.5.20 for values of $C^{\pm}_{jm}$)

--→ Note that even if $m = \pm j$ (and either $|j\,m+1\rangle$ or $|j\,m-1\rangle$ doesn't exist) then it still vanishes since $\hat{J}_+$ or $\hat{J}_-$ annihilates the state – i.e. $C^{\pm}_{jm}$ is zero.

Similarly, $\langle \hat{J}_y \rangle = \frac{1}{2i}\left[\underbrace{\langle \hat{J}_+ \rangle}_{0} - \underbrace{\langle \hat{J}_- \rangle}_{0}\right] = 0.$

2) $\langle \hat{J}_x^2 \rangle = \langle \hat{J}^2 - \hat{J}_y^2 - \hat{J}_z^2 \rangle$ But $\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle$ by symmetry,

so $2\langle \hat{J}_x^2 \rangle = \langle \hat{J}^2 - \hat{J}_z^2 \rangle$

$= \hbar^2 j(j+1) - \hbar^2 m^2$

$\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]$.

3) $\Delta J_x = \Delta J_y = \sqrt{\frac{\hbar^2}{2}[j(j+1) - m^2]} = \frac{\hbar}{\sqrt{2}}\sqrt{j^2 + j - m^2}$

$(\Delta J_x \Delta J_y)^2 = \frac{\hbar^4}{4}\left([j(j+1) - m^2]\right)^2 \overset{?}{\geq} |\langle \hat{J}_x \hat{J}_y \rangle|^2$

$$\langle \hat{J}_x \hat{J}_y \rangle = \frac{-i}{4} \langle (\hat{J}_+ + \hat{J}_-)(\hat{J}_+ - \hat{J}_-) \rangle$$

$$= \frac{-i}{4} \langle \hat{J}_+^2 + \hat{J}_-^2 + [\hat{J}_-, \hat{J}_+] \rangle$$

$$= \frac{-i}{4} \langle \hat{J}_+^2 + \hat{J}_-^2 - 2\hbar \hat{J}_z \rangle$$

$$= \frac{-i}{4} \left[ C^+_{jm+1} C^+_{jm} \underbrace{\langle jm | j\, m+2 \rangle}_{0} + C^-_{jm-1} C^-_{jm} \underbrace{\langle jm | j\, m-2 \rangle}_{0} - 2\hbar^2 m \right]$$

$$= \frac{i}{2} \hbar^2 m \quad \text{(Note that } \hat{J}_x \hat{J}_y \text{ is not Hermitian!)}$$

$$\text{so} \quad |\langle \hat{J}_x \hat{J}_y \rangle|^2 = \frac{\hbar^4}{4} m^2$$

Back to inequality:

$$\frac{\hbar^4}{4} \left( [j(j+1) - m^2] \right)^2 \overset{?}{\geq} \frac{\hbar^4}{4} m^2$$

$$j^2(j+1)^2 - 2m^2 j(j+1) + m^4 \overset{?}{\geq} m^2$$

Recall that $j \geq 0$ and $|m| \leq j$

so $j(j+1) \geq m^2$ and $j^2(j+1)^2 \geq m^4$

4)

If $j = \pm m$, $j^2 = m^2$:

$$j^2(j+1)^2 - 2j^3(j+1) + j^4 \overset{?}{=} j^2$$

$$j^4 + 2j^3 + \cancel{j^2} - \cancel{j^2} - 2j^4 - 2j^3 + j^4 \doteq 0$$

3 [12.6.1]

1) No angular dependence: $l = m = 0$.

2) Assume $V(r)$ vanishes at large $r$ (as stated).
With $l=0$, radial equation is $\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}\psi_E(r) = E\psi_E(r) - \underbrace{\cancel{V(r)}}_{0}\psi$

$$\Rightarrow A\left(\frac{-1}{a_0}\right)^2\left(\frac{-\hbar^2}{2\mu}e^{-r/a_0}\right) = EAe^{-r/a_0}$$

$$\Rightarrow E = \frac{-\hbar^2}{2\mu a_0^2}$$

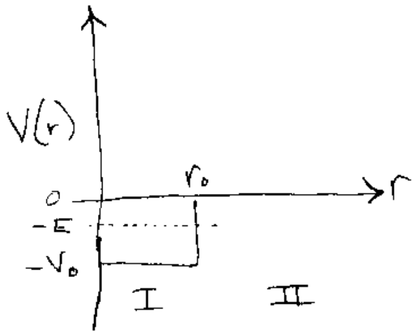
3) Back to radial Schr. eqn:

$$E\psi(r) + \frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\psi = V(r)\psi$$

$$\frac{\hbar^2}{2\mu}\cdot\frac{2}{r}\frac{d}{dr}\psi = V(r)\psi = \frac{\hbar^2}{\mu r}\left(\frac{-1}{a_0}\right)\psi$$

$$V(r) = \frac{-\hbar^2}{\mu a_0 r}$$

④ [12.6.9] — A little Bessel function practice.




Region I: $r\psi = j_0(k'r) = \frac{\sin k'r}{k'r}$

where $k' = \frac{\sqrt{2m(V_0-E)}}{\hbar}$

Region II (classically forbidden)

$$r\psi = \frac{e^{-\kappa r}}{\kappa r} \quad \text{where } \kappa = \frac{\sqrt{2mE}}{\hbar}$$

Match at $r_0$:

$$\underbrace{\frac{1}{j_0(k'r_0)} \frac{d}{dr}\Big|_{r_0} j_0(k'r_0)}_{\frac{k'r_0}{\sin k'r_0}\left(\frac{k'r_0}{\sin k'r_0} - \frac{\sin k'r_0}{k'r_0^2}\right)} = \underbrace{\frac{-\kappa r}{e^{-\kappa r}} \frac{d}{dr}\Big|_{r_0} \frac{-e^{-\kappa r}}{\kappa r}}_{= \frac{\kappa r_0}{e^{-\kappa r_0}}\left(-\kappa - \frac{1}{r_0}\right)\left(\frac{-e^{-\kappa r_0}}{\kappa r_0}\right)}$$

$$= k'\cot k'r_0 - \frac{1}{r_0} \qquad = -\kappa - \frac{1}{r_0}$$

$$\text{so } k'\cot k'r_0 - \cancel{\frac{1}{r_0}} = -\kappa - \cancel{\frac{1}{r_0}}$$

$$\frac{k'}{\kappa} = -\tan k'r_0$$

This is the same situation as 5.2.6 except the wave fn must vanish at $r \to 0$ so "even" solutions from 5.2.6 don't exist. At critical $V_0 = \frac{\pi^2\hbar^2}{8\mu r_0}$, solution has $E \to 0$: not bound.