HW5 errata

correction (12.2.3a) $L_z \xrightarrow{\text{coordinate basis}} x\left(-i\hbar\frac{\partial}{\partial y}\right)-y\left(-i\hbar\frac{\partial}{\partial x}\right)$

Transforming to polar coordinates:

$x=\rho\cos\phi \quad y=\rho\sin\phi,\ \tan\phi=\frac{y}{x}$

Now,

$\frac{\partial}{\partial x}=\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial y}=\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}$

Let's find $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}$

$$\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\tan\phi+\cancel{\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\tan\phi}=\frac{\partial}{\partial x}\frac{y}{x}$$
$$\frac{\partial\phi}{\partial x}\sec^2\phi=-\frac{y}{x^2}$$
$$\frac{\partial\phi}{\partial x}=-\frac{y}{x^2}\cos^2\phi$$
$$=-\frac{\rho\sin\phi}{\rho^2\cancel{\cos^2}\phi}\cancel{\cos^2\phi}$$
$$\rightarrow \frac{\partial\phi}{\partial x}=-\frac{\sin\phi}{\rho}$$

$$\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\tan\phi+\cancel{\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}\tan\phi}=\frac{\partial}{\partial y}\frac{y}{x}$$
$$\frac{\partial\phi}{\partial y}\sec^2\phi=\frac{1}{x}$$
$$\frac{\partial\phi}{\partial y}=\frac{\cos^2\phi}{\rho\cancel{\cos\phi}}$$
$$\rightarrow \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho}$$

$$\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\rho\cos\phi+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\rho\cos\phi=\frac{\partial}{\partial x}x$$
$$+\frac{\sin\phi}{\cancel{\rho}}\cancel{\rho}\sin\phi+\frac{\partial\rho}{\partial x}\cos\phi=1$$
$$\sin^2\phi+\frac{\partial\rho}{\partial x}\cos\phi=1$$
$$\rightarrow \frac{\partial\rho}{\partial x}=\cos\phi$$

$$\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\rho\sin\phi+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}\rho\sin\phi=\frac{\partial}{\partial y}y$$
$$\frac{\cos\phi}{\cancel{\rho}}\cancel{\rho}\cos\phi+\frac{\partial\rho}{\partial y}\sin\phi=1$$
$$\rightarrow \frac{\partial\rho}{\partial y}=\sin\phi$$

Putting it all together:

$$L_z\rightarrow\rho\cos\phi\left(-i\hbar\left(\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}+\sin\phi\frac{\partial}{\partial\rho}\right)\right)-\rho\sin\phi\left(-i\hbar\left(\frac{-\sin\phi}{\rho}\frac{\partial}{\partial\phi}+\cos\phi\frac{\partial}{\partial\rho}\right)\right)$$
$$=-i\hbar\cos^2\phi\frac{\partial}{\partial\phi}-i\hbar\cancel{\rho\cos\phi\sin\phi\frac{\partial}{\partial\rho}}-i\hbar\sin^2\phi\frac{\partial}{\partial\phi}+i\hbar\cancel{\rho\sin\phi\cos\phi\frac{\partial}{\partial\rho}}$$
$$=-i\hbar\left(\cos^2\phi\frac{\partial}{\partial\phi}+\sin^2\phi\frac{\partial}{\partial\phi}\right)$$

$$\boxed{L_z\rightarrow -i\hbar\frac{\partial}{\partial\phi}}$$

addition

12.2.3b) Under infinitesimal rotation $\varepsilon_z \vec{k}$, $\psi(x,y) = \psi(\rho, \phi)$ becomes $\psi(\rho, \phi - \varepsilon_z)$.

The finite rotation operator is:

$U[R(\phi_0 \vec{k})] = \lim_{N\to\infty} \left(I - \frac{i}{\hbar}\frac{\phi_0}{N} L_z\right)^N$ which in this case is

$\mathcal{U}[R(\varepsilon_z \vec{k})] = \left(I - \frac{i}{\hbar}\varepsilon_z L_z\right)$, for small ε_z.

so

$$\langle \rho, \phi | I - \tfrac{i}{\hbar}\varepsilon_z L_z | \psi \rangle = \psi(\rho, \phi - \varepsilon_z)$$

expanding the right hand side in Taylor series

$$\cancel{\langle \rho, \phi | \psi \rangle} - \frac{\varepsilon_z i}{\hbar}\langle \rho \phi | L_z | \psi \rangle = \cancel{\psi(\rho, \phi)} - \varepsilon_z \frac{\partial}{\partial \phi}\psi(\rho, \phi) + \cdots \underbrace{O(\varepsilon_z^2)}_{\text{small}}$$

$$\frac{i}{\hbar} L_z \psi(\rho, \phi) = -\frac{\partial}{\partial \phi}\psi(\rho, \phi)$$

$$\Rightarrow \boxed{L_z = -i\hbar \frac{\partial}{\partial \phi}}$$

Ulrian 12.4.1)(2) Let U_1, U_2, U_3 be three energy eigenfunctions of a single particle in some potential. Construct $\psi_A(x_1 x_2, x_3)$ of three fermions in this potential one of which is in U_1 one in U_2 and one in U_3, using ε_{ijk} tensor.

$$\sum |x_i x_j x_k\rangle \varepsilon_{ijk} = |x_1 x_2 x_3\rangle - |x_1 x_3 x_2\rangle$$

$$+ |x_2 x_3 x_1\rangle - |x_2 x_1 x_3\rangle + |x_3 x_1 x_2\rangle - |x_3 x_2 x_1\rangle$$