P4410 PROBLEM SET 5 SOLUTIONS

① 11.2.1

$$\langle P\rangle = \langle\psi|\hat{P}|\psi\rangle$$

$$= \int dx'\int dx'' \langle\psi|x'\rangle\langle x'|\hat{P}|x''\rangle\langle x''|\psi\rangle$$

$$= \int dx'\int dx''\, \psi^*(x')\,(-i\hbar)\,\delta(x'-x'')\frac{d}{dx''}\psi(x'') = -i\hbar\int dx'\,\psi^*(x')\frac{d}{dx'}\psi(x')$$

$$\langle\psi|\hat{T}^\dagger\hat{P}\hat{T}|\psi\rangle = \int dx'\int dx''\langle\psi|x'\rangle\langle x'|\hat{T}^\dagger\,\hat{P}\,\hat{T}^\dagger|x''\rangle\langle x''|\psi\rangle$$

$$= \int dx'\int dx''\,\psi^*(x')\langle x'+\varepsilon|e^{-i\varepsilon g(x')/\hbar}\,\hat{P}\,e^{i\varepsilon g(x'')/\hbar}|x''+\varepsilon\rangle\quad\psi(x'')$$

$$= \int dx'\int dx''\,\psi^*(x')e^{-i\varepsilon g(x')/\hbar}(-i\hbar)\,\delta\big((x'+\varepsilon)-(x''+\varepsilon)\big)\frac{d}{dx''}\,e^{i\varepsilon g(x'')/\hbar}\,\psi(x'')$$

$$= \int dx'\quad \psi^*(x')e^{-i\varepsilon g(x')/\hbar}(-i\hbar)\frac{d}{dx'}\,e^{i\varepsilon g(x')/\hbar}\,\psi(x')$$

$$= \int dx'\,\psi^*(x')\cancel{e^{-i\varepsilon g(x')/\hbar}}(-i\hbar)\left\{\left[\frac{i\varepsilon}{\hbar}\frac{dg(x')}{dx'}\cancel{e^{i\varepsilon g(x')/\hbar}}\psi(x')\right]+\left[\cancel{e^{i\varepsilon g(x')/\hbar}}\frac{d\psi(x')}{dx'}\right]\right\}$$

$$= \left(-i\hbar\int dx'\,\psi^*(x')\,\frac{i\varepsilon}{\hbar}\,f(x')\,\psi(x')\right) + \left(-i\hbar\int dx'\,\psi^*(x')\frac{d}{dx'}\psi(x')\right)$$

$$= \varepsilon\langle f(x)\rangle + \langle P\rangle \qquad \text{QED}$$

(2) 11.4.1 $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$

Recall that $i\hbar \frac{d\hat{U}}{dt} = \hat{H}\hat{U} \Longrightarrow [\hat{\mathbb{P}}, \hat{U}(t)] = 0$ if $[\hat{\mathbb{P}}, \hat{H}] = 0$

so, if $\hat{\mathbb{P}}|\psi(0)\rangle = \pm|\psi(0)\rangle$ then

$$\begin{aligned}\hat{\mathbb{P}}|\psi(t)\rangle &= \hat{\mathbb{P}}\hat{U}|\psi(0)\rangle \\ &= \hat{U}\hat{\mathbb{P}}|\psi(0)\rangle \\ &= \pm\hat{U}|\psi(0)\rangle \\ &= \pm|\psi(t)\rangle\end{aligned}$$

QED $\Longrightarrow$ Parity of state is unchanged.

11.4.2) Particle in potential $V(x) = V_0 \sin\left(\frac{2\pi x}{a}\right)$

Is momentum conserved?

$$i\hbar \frac{d}{dt}\langle P\rangle = \langle [P, H] \rangle$$

$$= \left\langle \left[P, \frac{P^2}{2m} + V(x)\right] \right\rangle$$

$$= \left\langle \left[P, \frac{P^2}{2m}\right] + [P, V(x)] \right\rangle \quad \text{let } |\omega\rangle \text{ be test state}$$

$$= \left\langle 0 + \frac{\hbar}{i}\frac{d}{dx}\left(V_0 \sin\frac{2\pi x}{a}\right)|\omega\rangle - \left(V_0 \sin\frac{2\pi x}{a}\right)\frac{\hbar}{i}\frac{d}{dx}|\omega\rangle \right\rangle$$

$$= \frac{\hbar}{i} V_0 \left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi x}{a}\right)|\omega\rangle + \frac{\hbar}{i}V_0 \sin\left(\frac{2\pi x}{a}\right)\frac{d}{dx}|\omega\rangle - \cancel{V_0 \sin\frac{2\pi x}{a}\frac{\hbar}{i}\frac{d}{dx}|\omega\rangle} \rangle$$

$$= \frac{\hbar}{i} V_0 \frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right) \neq 0$$

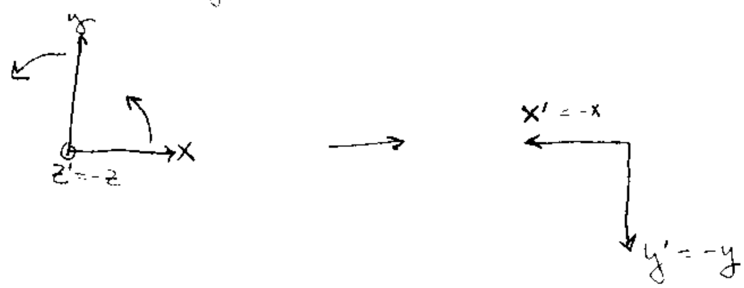
Not conserved.

II.4.3 In a certain reaction an electron comes out with its spin always parallel to its momentum. If we look at this reaction under parity the spin changes direction but the direction of the emitted particle does not. That is, the spin and the emitted particle are antiparallel under parity. But we are told this process never happens. Thus parity is violated.

⑤ <u>11.4.4</u>

First, reflect about x-y plane: $(x,y,z) \to (x,y,-z)$

Now rotate by π about z axis:



so $(x,y,-z) \to (-x,-y,-z)$ i.e. the same as $\hat{\mathbb{P}}(x,y,z)$.

⑥ 12.3.4 Express azimuthal part of wave function as:

$$\left(\frac{\rho}{\Delta}\cos\phi + \sin\phi\right) = \left[\frac{\rho}{2\Delta}\left(e^{i\phi}+e^{-i\phi}\right) + \frac{1}{2i}\left(e^{i\phi}-e^{-i\phi}\right)\right]$$

$$= e^{i\phi}\underbrace{\left(\frac{\rho}{2\Delta}-\frac{i}{2}\right)}_{=C^*} + e^{-i\phi}\underbrace{\left(\frac{\rho}{2\Delta}+\frac{i}{2}\right)}_{=C}$$

$$\langle\phi|\psi\rangle = C^*\langle\phi|m=1\rangle + C\langle\phi|m=-1\rangle$$

$$\Longrightarrow |\psi\rangle = C^*|1\rangle + C|-1\rangle$$

so $P(m=1) = \dfrac{|C^*|^2}{|C^*|^2+|C|^2} = \dfrac{1}{2}$

$P(m=-1) = \dfrac{|C|^2}{|C^*|^2+|C|^2} = \dfrac{1}{2}$

12.2.3) $L_z \xrightarrow[\text{basis}]{\text{coordinate}} x\left(-i\hbar \frac{\partial}{\partial y}\right) - y\left(-i\hbar \frac{\partial}{\partial x}\right)$

Transforming to polar coordinates:

$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad \tan\phi = \frac{y}{x}$

$$\frac{\partial}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}$$

so, $\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\tan\phi = \frac{\partial}{\partial x}\frac{y}{x}$ and $\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\tan\phi = \frac{\partial}{\partial y}\frac{y}{x}$

$$\frac{\partial\phi}{\partial x}\sec^2\phi = \frac{y}{-x^2} \quad \text{so} \quad \frac{\partial\phi}{\partial y}\sec^2\phi = \frac{1}{x}$$

$$\frac{\partial\phi}{\partial x} = \frac{y\cos^2}{-x^2} \qquad \frac{\partial\phi}{\partial y} = \frac{\cos^2\phi}{x}$$

Now plugging in,

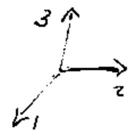
$$L_z \rightarrow x\left(-i\hbar\frac{\cos^2\phi}{x}\frac{\partial}{\partial\phi}\right) - y\left(+i\hbar\frac{y\cos^2\phi}{x^2}\frac{\partial}{\partial\phi}\right)$$

$$= \cos^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right) - \frac{y^2}{x^2}\cos^2\left(+i\hbar\frac{\partial}{\partial\phi}\right)$$

$$= \cos^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \sin^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right)$$

$$= \left(-i\hbar\frac{\partial}{\partial\phi}\right)$$

12.4.1 $\vec{c} = \vec{a} \times \vec{b}$, Eq (12.4.8)



$$\vec{c} = \begin{vmatrix} 1 & 2 & 3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\boxed{\vec{c} = (a_2 b_3 - a_3 b_2)1 + (a_3 b_1 - a_1 b_3)2 + (a_1 b_2 - a_2 b_1)3}$$

And,

$$c_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k, \quad Eq\ (12.4.9)$$

$$= \sum_{j=1}^{3} \left(\varepsilon_{ij1} a_j b_1 + \varepsilon_{ij2} a_j b_2 + \varepsilon_{ij3} a_j b_3 \right)$$

$$= \varepsilon_{i11} a_1 b_1 + \underset{-1}{\varepsilon_{i21}} a_2 b_1 + \underset{+}{\varepsilon_{i31}} a_3 b_1 + \underset{+}{\varepsilon_{i12}} a_1 b_2 + \varepsilon_{i22} a_2 b_2$$

$$+ \underset{-1}{\varepsilon_{i32}} a_3 b_2 + \underset{-1}{\varepsilon_{i13}} a_1 b_3 + \underset{+}{\varepsilon_{i23}} a_2 b_3 + \cancel{\varepsilon_{i33} a_3 b_3}$$

$\Rightarrow$ $c_1 = a_2 b_3 - a_3 b_2$, $c_2 = a_3 b_1 - a_1 b_3$, $c_3 = a_1 b_2 - a_2 b_1$

$$\vec{c} = c_1 1 + c_2 2 + c_3 3$$

$$\boxed{\vec{c} = (a_2 b_3 - a_3 b_2)1 + (a_3 b_1 - a_1 b_3)2 + (a_1 b_2 - a_2 b_1)3}$$

Thus Eq (12.4.8) is equivalent to Eq (12.4.9)