P4410 | PROBLEM SET 4 | SOLUTIONS

Shankar

10.3.2

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{3}}\left\{ |3,3,4\rangle + |3,4,3\rangle + |4,3,3\rangle \right\}}$$

Liboff

12.20

$$\hat{\chi}_\pm = \frac{1}{\sqrt{2}}\left\{ \mathbb{1} \pm \hat{\chi} \right\}$$

$$\boxed{\begin{aligned} \hat{\chi}_+ \varphi(1,2) &= \frac{1}{\sqrt{2}}\left(\varphi(1,2) + \varphi(2,1)\right) = \varphi_S(1,2) \\ \hat{\chi}_- \varphi(1,2) &= \frac{1}{\sqrt{2}}\left(\varphi(1,2) - \varphi(2,1)\right) = \varphi_A(1,2) \end{aligned}}$$

11.73

a) $\boxed{\hat{\rho} = \frac{1}{2}|E_1\rangle\langle E_1| + \frac{1}{2}|E_5\rangle\langle E_5|}$; $\langle x|E_i\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{i\pi}{L}x\right)$

b) $|\psi_1\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}|E_5\rangle$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}|E_1\rangle - \frac{1}{\sqrt{2}}|E_5\rangle$

$$\boxed{\hat{\rho} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|}$$

$$= \frac{1}{2}|E_1\rangle\langle E_1| + \frac{1}{2}|E_5\rangle\langle E_5|$$

② 10.3.5

(part 1) Let $|p\rangle$ be an eigenstate of $\hat{P}_{12}$ with eigenvalue p.

$$\hat{P}_{12}|p\rangle = p|p\rangle$$

$$\hat{P}_{12}^2|p\rangle = p^2|p\rangle \quad \text{but } \hat{P}_{12}^2 = \mathbb{1}, \quad \text{so } p^2 = 1$$

$$\Longrightarrow \underline{p = \pm 1.}$$

(Part 2) Expand $|\omega_1\omega_2\rangle$ in $|x_1x_2\rangle$ basis:

$$|\omega_1\omega_2\rangle = \int dx_1 \int dx_2 \langle x_1 x_2|\omega_1\omega_2\rangle |x_1x_2\rangle = \int dx_1\, dx_2\, \omega_{12}(x_1,x_2)|x_1x_2\rangle$$

$$\hat{P}_{12}|\omega_1\omega_2\rangle = \int dx_1 \int dx_2\, \omega_{12}(x_1,x_2)|x_2x_1\rangle$$

Rename integration variables $x_1 \leftrightarrow x_2$

$$= \int dx_2 \int dx_1\, \omega_{12}(x_2,x_1)|x_1x_2\rangle$$

$$= \int dx_2 \int dx_1 \langle x_2 x_1|\omega_1\omega_2\rangle |x_1x_2\rangle$$

but
$$\langle x_2x_1|\omega_1\omega_2\rangle = \left(\langle x_2|^{(1)} \otimes \langle x_1|^{(2)}\right)\left(|\omega_1\rangle^{(1)} \otimes |\omega_2\rangle^{(2)}\right)$$

$$= \langle x_2|\omega_1\rangle\langle x_1|\omega_2\rangle$$

$$= \langle x_1|\omega_2\rangle\langle x_2|\omega_1\rangle$$

$$= \langle x_1x_2|\omega_2\omega_1\rangle$$

So
$$P_{12}|\omega_1\omega_2\rangle = \int dx_2 \int dx_1 \langle x_1x_2|\omega_2\omega_1\rangle |x_1,x_2\rangle = |\omega_2\omega_1\rangle$$

part 3 Show identities by using action on $|\psi\rangle$:

$$\hat{P}_{12}\hat{X}_1\hat{P}_{12}|\psi\rangle = \int dx_1 \int dx_2\, \hat{P}_{12}\hat{X}\hat{P}_{12}|x_1x_2\rangle\langle x_1x_2|\psi\rangle$$

$$= \int dx_1 \int dx_2\, \hat{P}_{12}\hat{X}_1\hat{P}_{12}|x_1x_2\rangle \psi(x_1,x_2)$$

$$= \int dx_1 \int dx_2\, \hat{P}_{12}\hat{X}_1|x_2x_1\rangle \psi(x_1,x_2)$$

$$= \int dx_1 \int dx_2\, x_2\hat{P}_{12}|x_2x_1\rangle \psi(x_1,x_2)$$

$$= \int dx_1 \int dx_2\, x_2|x_1x_2\rangle \psi(x_1,x_2)$$

$$= \int dx_1 \int dx_2\, \hat{X}_2|x_1x_2\rangle \psi(x_1,x_2)$$

$$= \int dx_1 \int dx_2\, \hat{X}_2|x_1x_2\rangle\langle x_1x_2|\psi\rangle$$

$$= \hat{X}_2|\psi\rangle$$

To show for $\hat{P}_{12}\hat{X}_2\hat{P}_{12} = \hat{X}_1$, switch $1 \leftrightarrow 2$ above.

To show for $\hat{P}_{12}\hat{p}_1\hat{P}_{12} = \hat{p}_2$, show as above but expand in p_1, p_2 eigenstates instead.

Assume $\hat{\Omega} = \sum_{mnkl} a_{mnkl}\hat{X}_1^m\hat{X}_2^n\hat{p}_1^k\hat{p}_2^l$

$$\hat{P}_{12}\hat{\Omega}\hat{P}_{21} = \sum_{mnkl} a_{mnkl}\hat{P}_{12}X_1^m X_2^n p_1^k p_2^l \hat{P}_{21}$$

Now insert $\hat{P}_{12}\hat{P}_{12} = 1$ between each pair of operators:

$$= \sum_{mnkl} a_{mnkl}\underbrace{P_{12}X_1^m P_{12}}_{=(P_{12}\hat{X}_1P_{12})(P_{12}\hat{X}_1P_{12})(\cdots) = (\hat{P}_{12}\hat{X}_1\hat{P}_{12})^m = \hat{X}_2^m} P_{12}X_2^n P_{12}P_{12}p_1^k P_{12}P_{12}p_2^l P_{12}$$

so $\hat{P}_{12}\hat{\Omega}\hat{P}_{21} = \sum_{mnkl} a_{mnkl}\, X_2^m X_1^n p_2^k p_1^l$

$$= \hat{\Omega}\left(\hat{X}_2, \hat{X}_1, \hat{p}_2, \hat{p}_1\right) \qquad \textbf{QED}.$$

part 4 $\hat{H} = T_1 + T_2 + \underbrace{V(x_1, x_2)}_{= V(x_2, x_1)}$ because particles identical.

so $\hat{P}_{12}\hat{H}\hat{P}_{12} = P_{12}\left(\hat{T}_1 + \hat{T}_2 + \hat{V}(x_1, x_2)\right)\hat{P}_{12}$

$$= \hat{T}_2 + \hat{T}_1 + \hat{V}(x_2, x_1) = \hat{H}.$$

$$\hat{U} = e^{-\frac{i\hat{H}t}{\hbar}} = \sum_m \frac{1}{m!}\left(\frac{-i\hat{H}t}{\hbar}\right)^m$$

so $\hat{P}_{12}\hat{U}\hat{P}_{12} = \sum_m \frac{1}{m!}\left(\frac{-it}{\hbar}\right)^m \underbrace{\hat{P}_{12}\hat{H}^m\hat{P}_{12}}_{= \hat{H}^m}$

$$= \sum_m \frac{1}{m!}\left(\frac{-i\hat{H}t}{\hbar}\right)^m = \hat{U}.$$

Now take $|P_{12}^{\pm}\rangle$ such that $\hat{P}_{12}|P_{12}^{\pm}\rangle = \pm|P_{12}^{\pm}\rangle$

$\hat{P}_{12}\,|P_{12}^{\pm}(t)\rangle = \hat{P}_{12}\hat{U}\,|P_{12}^{\pm}(0)\rangle$

$$= \hat{P}_{12}\hat{P}_{12}\hat{U}\hat{P}_{12}|P_{12}^{\pm}(0)\rangle$$

$$= \hat{U}\hat{P}_{12}|P_{12}^{\pm}(0)\rangle$$

$$= \pm\hat{U}|P_{12}^{\pm}(0)\rangle$$

$$= \pm|P_{12}^{\pm}(t)\rangle.$$

(4) $\hat{\rho} = \sum_i w_i |i\rangle\langle i|$ $\longrightarrow$ evolve $|i\rangle$, $\langle i|$ in time.

so $\hat{\rho}(t) = \sum_i w_i \hat{U}(t)|i\rangle\langle i|\hat{U}^\dagger(t)$

$$= \hat{U}(t) \sum_i w_i |i\rangle\langle i| \hat{U}^\dagger(t)$$

$$= \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$$

(5) A pure ensemble has one $w_i = 1$, all others 0.

So $\hat{\rho}(0) = |i\rangle\langle i|$; $\hat{\rho}^2 = |i\rangle \underbrace{\cancel{\langle i|i\rangle}}_{1} \langle i| = |i\rangle\langle i| = \hat{\rho}$

$\hat{\rho}(t) = \hat{U}(t)|i\rangle\langle i|\hat{U}^\dagger(t)$

$\hat{\rho}^2(t) = \hat{U}(t)|i\rangle\langle i|\underbrace{\cancel{\hat{U}^\dagger(t)\hat{U}(t)}}_{1}|i\rangle\langle i|\hat{U}^\dagger(t)$

$$= \hat{U}(t)|i\rangle \underbrace{\cancel{\langle i|i\rangle}}_{1} \langle i|\hat{U}^\dagger(t)$$

$$= \hat{U}(t)|i\rangle\langle i|\hat{U}^\dagger(t) = \rho(t)$$

so if $\hat{\rho}^2(0) = \rho(0)$, then $\hat{\rho}^2(t) = \hat{\rho}(t)$.