1

P4410 PROBLEM SET 3 SOLUTIONS

Shankar 4.2.1

(1) The possible values one can obtain if L_z is measured are its eigenvalues.

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \boxed{\text{Eigenvalues are } \{-1, 0, 1\}}$$

(2) The state in which $L_z = 1$ is the corresponding eigenvector

$$|L_z; 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In $|L_z; 1\rangle$, What is $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

$$\langle L_x \rangle_{z1} = \langle L_z; 1 | L_x | L_z; 1 \rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \boxed{0 = \langle L_x \rangle_{z,1}}$$

$$\langle L_x^2 \rangle_{z,1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{1/2 = \langle L_x^2 \rangle_{z,1}}$$

$$\Delta L_x = \left[\langle L_z; 1 | (L_x - \cancelto{0}{\langle L_x \rangle})^2 | L_z; 1 \rangle \right]^{1/2}$$

$$\boxed{\Delta L_x = \frac{1}{\sqrt{2}}}$$

Shankar 4.2.1 contined

(3) Find normalized eigenstates and the eigenvalues of L_x in L_z basis

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues are $\{-1, 0, 1\}$

The eigenvectors are $\left\{ \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\}$

and normalizing

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\}$$

Writing in terms of L_z basis:

-1: $\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

0: $-\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$+1$: $\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

4.2.1 cont'd

(4) Particle is in state $|L_z=-1\rangle \Longleftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$P(L_x=-1)=\left|\langle L_x=-1|L_z=-1\rangle\right|^2=\left|\begin{pmatrix}\frac{1}{2} & \frac{-\sqrt{2}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{4}$$

$$P(L_x=0)=\left|\langle L_x=0|L_z=-1\rangle\right|^2=\left|\begin{pmatrix}\frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{2}$$

$$P(L_x=-1)=\left|\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{4}$$

(5) If $L_z^2=1$, $L_z=\pm 1$. If this is an ideal measurement, it leaves the particle in a state where the $L_z=0$ component has been "knocked out."

$|\psi\rangle \longrightarrow \begin{pmatrix}1/2\\0\\\sqrt{1/2}\end{pmatrix}$ But this needs to be normalized, so

the state becomes $\begin{pmatrix}1/2\\0\\\sqrt{1/2}\end{pmatrix}\cdot\left[\begin{pmatrix}\frac{1}{2} & 0 & \sqrt{\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\0\\\sqrt{\frac{1}{2}}\end{pmatrix}\right]^{-1/2}=\frac{2}{\sqrt{3}}\begin{pmatrix}\frac{1}{2}\\0\\\sqrt{\frac{1}{2}}\end{pmatrix}=\boxed{\begin{pmatrix}\sqrt{\frac{1}{3}}\\0\\\sqrt{\frac{2}{3}}\end{pmatrix}}$

Probability of this is $P(L_x=1)+P(L_x=-1)=\frac{1}{4}+\frac{1}{2}=\boxed{\frac{3}{4}}$

(6) The overall phase is irrelevant physically, but the relative phase of the three terms is important.

(2) Shankar 4.2.3

In x basis, $\langle p \rangle = \int dx\, \psi^*(x) \, -i\hbar \frac{d}{dx} \psi(x)$

so $-i\hbar \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} \frac{d}{dx} \psi(x) e^{\frac{+ip_0 x}{\hbar}}$

$$= -i\hbar \int dx\, \psi^*(x)\, e^{\frac{-ip_0 x}{\hbar}} \left[\frac{+ip_0}{\hbar} \psi(x)\, e^{\frac{+ip_0 x}{\hbar}} + e^{\frac{+ip_0 x}{\hbar}} \frac{d}{dx} \psi(x) \right]$$

$$= -i\hbar \left[\int dx \left(\frac{ip_0}{\hbar} \cancel{e^{\frac{-ip_0 x}{\hbar}}} \psi^*(x) \cancel{e^{\frac{ip_0 x}{\hbar}}} \psi(x) \right) + \int dx \left(\psi^*(x) \cancel{e^{\frac{-ip_0 x}{\hbar}}} \cancel{e^{\frac{ip_0 x}{\hbar}}} \frac{d}{dx} \psi(x) \right) \right]$$

$$= p_0 \underbrace{\int dx\, \psi^*(x) \psi(x)}_{=1} - \underbrace{i\hbar \int dx\, \psi^*(x) \frac{d}{dx} \psi(x)}_{= \langle p \rangle}$$

$$= p_0 + \langle p \rangle \qquad \text{QED.}$$

(3) Shankar 5.1.1

$E = \frac{p^2}{2m} \Rightarrow p = \alpha\sqrt{2mE}$; clearly $|p\rangle = |E, \alpha\rangle$.

$dp = \frac{\alpha m}{\sqrt{2mE}} dE$

so: $\int_{-\infty}^{\infty} dp\, |p\rangle\langle p| e^{-iEt/\hbar}$

$$= \int_{-\infty}^{0} dp\, |p\rangle\langle p| e^{-iEt/\hbar} + \int_{0}^{\infty} dp\, |p\rangle\langle p| e^{-iEt/\hbar}$$ Change integration variable:

$$= -\int_{0}^{+\infty} dE \frac{-m}{\sqrt{2mE}} |E,-1\rangle\langle E,-1| e^{\frac{-iEt}{\hbar}} + \int_{0}^{\infty} dE \frac{m}{\sqrt{2mE}} |E,+1\rangle\langle E,+1| e^{\frac{-iEt}{\hbar}} = \sum_{\alpha=\pm1} \int_{0}^{\infty} dE \frac{m}{\sqrt{2mE}} |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar}$$

QED.

④ Shankar 5.2.1

Let $|1\rangle$ = original ground state

$|1'\rangle$ = ground state of new well.

If well extends from $-\frac{L}{2}$ to $\frac{L}{2}$ before,

$-L$ to L after

then
$$\langle x|1\rangle = \begin{cases} \sqrt{\frac{2}{L}} \cos\frac{\pi x}{L} & |x|<\frac{L}{2} \\ 0 & |x|>\frac{L}{2} \end{cases}$$

$$\langle x|1'\rangle = \begin{cases} \sqrt{\frac{1}{L}} \cos\frac{\pi x}{2L} & |x|<L \\ 0 & |x|>L \end{cases}$$

Prob. of finding particle in $|1'\rangle$ is $|\langle 1|1'\rangle|^2$

$$= \left| \int_{-\infty}^{\infty} dx \, \langle 1|x\rangle\langle x|1'\rangle \right|^2 = \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \frac{\sqrt{2}}{L} \cos\frac{\pi x}{L} \cos\frac{\pi x}{2L} \right|^2$$

Let $\xi = \frac{\pi x}{2L}$

$dx = \frac{2L}{\pi} d\xi$

$$= \left| \int_{-\pi/4}^{\pi/4} d\xi \frac{2L\sqrt{2}}{\pi L} \cos 2\xi \cos\xi \right|^2$$

$$= \frac{8}{\pi^2} \left| \int_{-\pi/4}^{\pi/4} d\xi \cos 2\xi \cos\xi \right|^2$$

$$= \frac{8}{\pi^2} \left| \frac{\sqrt{8}}{3} \right|^2 = \left(\frac{8}{3\pi} \right)^2$$

(5) Shankar 5.22 part (a)

(a) Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, Where E_0 is the lowest energy eigenvalue

Expand $|\psi\rangle$ in the eigenbasis of H

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n |\psi_n\rangle \,, \quad H|\psi_n\rangle = E_n|\psi_n\rangle$$

Then $$H|\psi\rangle = \sum_{n=0}^{\infty} a_n H|\psi_n\rangle = \sum_{n=0}^{\infty} a_n E_n |\psi_n\rangle$$

Taking the inner product with $\langle\psi|$

$$\langle\psi|H|\psi\rangle = \sum_{n=0}^{\infty} a_n E_n \langle\psi|\psi_n\rangle$$

$$= \sum_{n=0}^{\infty} a_n^2 E_n = a_0^2 E_0 + a_1^2 E_1 + a_2^2 E_2 + \cdots$$

If E_0 is the lowest energy eigenvalue...

$E_0 \leq E_n$, for $n > 0$

Then we can shrink all $E_n, n>0$ to E_0 and we have the inequality

$$\langle\psi|H|\psi\rangle \geq E_0 \Big(\underbrace{\sum_{n=0}^{\infty} a_n^2}_{1 \text{ since } |\psi\rangle \text{ is normalized}}\Big)$$

$$\Rightarrow \langle\psi|H|\psi\rangle \geq E_0$$

6

10.1.1

(1) $[\Omega_1^{(1)} \otimes I^{(2)}, I^{(1)} \otimes \Lambda_2^{(2)}] \underbrace{|\omega_1\rangle \otimes |\omega_2\rangle}_{\text{test state}}$

$= \Omega_1^{(1)} \otimes I^{(2)} (I^{(1)} \otimes \Lambda_2^{(2)}) |\omega_1\rangle \otimes |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} \otimes I^{(2)}) |\omega_1\rangle \otimes |\omega_2\rangle$

$= \Omega_1^{(1)} \otimes I^{(2)} (I^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} |\omega_1\rangle \otimes I^{(2)} |\omega_2\rangle$

$= (\Omega_1^{(1)} \otimes I^{2}) |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} |\omega_1\rangle \otimes |\omega_2\rangle$

$= \Omega_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle - \Omega_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{2} |\omega_2\rangle$
$= 0$ ✓

(2) $(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\theta_1^{(1)} \otimes \Lambda_2^{(2)}) \underbrace{|\omega_1\rangle \otimes |\omega_2\rangle}_{\text{test state}}$

$(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\theta_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle)$

$\Omega_1^{(1)} \theta_1^{(1)} |\omega_1\rangle \otimes \Gamma_2^{(2)} \Lambda_2^{(2)} |\omega_2\rangle$

$= (\Omega \ \theta)_1^{(1)} \otimes (\Gamma \ \Lambda)_2^{(2)}$

(3) If $[\Omega_1^{(1)}, \Lambda_1^{(1)}] = \Gamma_1^{(1)}$

Show

$$[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}] = \Gamma_1^{(1)} \otimes I^{(2)}$$

$$\left((\Omega_1^{(1)} \otimes I^{(2)})(\Lambda_1^{(1)} \otimes I^{(2)}) - (\Lambda_1^{(1)} \otimes I^{(2)})(\Omega_1^{(1)} \otimes I^{(2)})\right)|\omega_1\rangle \otimes |\omega_2\rangle$$

$$= (\Omega_1^{(1)}\Lambda_1^{(1)} - \Lambda_1 \Omega_1^{(1)})|\omega_1\rangle \otimes I^{(2)}|\omega_2\rangle$$

By the given commutation relation:

$$= (\Gamma_1^{(1)} \otimes I^{(2)})|\omega_1\rangle \otimes |\omega_2\rangle \checkmark$$

(4)

$$(\Omega_1^{(1)\times(2)} + \Omega_2^{(1)\times(2)})^2$$

$$= (\Omega^{2\,(1)\times(2)} + \Omega_1^{(1)\times(2)}\Omega_2^{(1)\times(2)} + \Omega_2^{(1)\times(2)}\Omega_1^{(1)\times(2)} - \Omega_2^{2\,(1)\times(2)})|\omega_1\rangle \otimes |\omega_2\rangle$$

$$= (\Omega^2)^{(1)}|\omega_1\rangle \otimes I^{(2)}|\omega_2\rangle + \Omega_1^{(1)}|\omega_1\rangle \otimes \Omega_2^{(2)}|\omega_2\rangle + \Omega_1^{(1)}|\omega_1\rangle \otimes \Omega_2^{(2)}|\omega_2\rangle + I^{(1)}|\omega_1\rangle \otimes (\Omega_2^2)^{(2)}$$

$$= (\Omega_1^2)^{(1)} \otimes I^{(2)} + I^{(1)} \otimes (\Omega_2^2)^{(2)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)} \checkmark$$