P4410 | PROBLEM SET 2 | SOLUTIONS

(2) (S.1.9.2)

Check that $\hat{U}^\dagger \hat{U} = 1$: $\hat{U} = e^{i\hat{H}}$, $\hat{U}^\dagger = e^{-i\hat{H}^\dagger} = e^{-i\hat{H}}$

$$\hat{U}^\dagger \hat{U} = e^{-i\hat{H} + i\hat{H}} = 1.$$

(3) (1.10.1)

$$\int dx\, \delta(ax) = \frac{\pm 1}{a} \int dy\, \delta(y) \quad \text{where } y = ax$$

(...where "±" is as needed to keep the result positive)

$$= \pm \frac{1}{a} = \frac{1}{|a|}$$

Similarly we could take $\int dx\, \delta(ax) f(x)$ and find that equal to $\frac{f(0)}{|a|}$. So $\delta(ax)$ behaves as $\frac{\delta(x)}{|a|}$.

(4) (1.10.3)

$$\int_{-\infty}^{x'} dx\, \delta(x) = \begin{cases} 0 & x' < 0 \\ 1 & x' > 0 \\ \text{Undefined} & x' = 0 \end{cases}$$

...which seems to qualify as a θ function!

Solutions to Shankar 1.8.10 and 1.10.4
By
Michael Allman

1.8.10

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

First compute the commutator

$$\begin{aligned} \Omega\Lambda - \Lambda\Omega &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \qquad \textbf{(1.1)}$$

Since Λ is non-degenerate, theorem 13 says that any eigenvector of Λ is also an eigenvector of Ω. Thus we just need to find the eigenvectors of Λ to simultaneously diagonalize both operators.

$$\begin{aligned} \det[\Lambda - \lambda I] &= \det\begin{pmatrix} (2-\lambda) & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & (2-\lambda) \end{pmatrix} \\ &= (2-\lambda)(\lambda^2 - 2\lambda - 3) \\ &= (2-\lambda)(\lambda-3)(\lambda+1) \\ &= 0 \end{aligned} \qquad \textbf{(1.2)}$$

Thus the eigenvalues are

$$\lambda = -1, 2, 3 \qquad \textbf{(1.3)}$$

Now find the corresponding eigenvectors

$\underline{\lambda = -1}$:

$$\Lambda|v\rangle = -|v\rangle \Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -k_1 \\ -k_2 \\ -k_3 \end{pmatrix}$$

$$\begin{aligned} &\Rightarrow 2k_1 + k_2 + k_3 = -k_1 \\ &\quad k_1 \quad - k_3 = -k_2 \\ &\Rightarrow k_1 = -s \\ &\quad k_2 = 2s \\ &\quad k_3 = s \end{aligned} \tag{1.4}$$

Choose s such that the eigenvector is normalized (we don't have to, its just good practice to *always* work with orthonormal bases and since these are eigenvectors of a Hermitian operator corresponding to non-degenerate eigenvalues, the eigenvectors are automatically orthogonal). Thus,

$$|v\rangle_{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \tag{1.5}$$

Following the same procedure for $\lambda = 2,3$, I get

$$|v\rangle_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \; and \; |v\rangle_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.6}$$

Now the unitary matrix that diagonaizes both operators is given by

$$U = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.7}$$

Finally check to make sure this is correct

$$U^{\dagger}\Omega U = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$U^{\dagger}\Lambda U = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \textbf{(1.8)}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

1.10.4

$$\Psi(x,t) = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \cos\omega_m t \langle m|\Psi(0)\rangle$$

$$\langle m|\Psi(0)\rangle = \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \Psi(x,0) dx$$

$$= \sqrt{\frac{2}{L}} \left[\int_0^{L/2} \frac{2h}{L} x \sin\left(\frac{m\pi x}{L}\right) dx + \int_{L/2}^{L} \frac{2h}{L}(L-x) \sin\left(\frac{m\pi x}{L}\right) dx \right] \quad \textbf{(1.9)}$$

$$= \frac{2\sqrt{2L}h}{(m\pi)^2} \left[2\sin\left(\frac{m\pi}{2}\right) - \sin(m\pi) \right]$$

Thus we have

$$\begin{aligned}\Psi(x,t) &= \sum_{m=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\frac{2\sqrt{2L}h}{(m\pi)^2}\left(2\sin\left(\frac{m\pi}{2}\right)-\sin(m\pi)\right)\\ &= \sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\frac{4h}{(m\pi)^2}\left(2\sin\left(\frac{m\pi}{2}\right)-\sin(m\pi)\right)\end{aligned} \qquad \textbf{(1.10)}$$

And now take note that

$$\sin(m\pi) = 0 \quad \forall m \in \{1,2,...\} \qquad \textbf{(1.11)}$$

Thus we are left with

$$\Psi(x,t) = \sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\frac{8h}{(m\pi)^2}\sin\left(\frac{m\pi}{2}\right) \qquad \textbf{(1.12)}$$