Selected Solutions
Chapter 1, Shankar

1.1.3 Do functions that vanish at $x=0$, $x=L$ form a vector space?

Yes.

Do periodic functions obeying $f(0)=f(L)$ form a vector space?

Yes.

Functions that obey $f(0)=4$?
No. They are not closed under scalar multiplication.

(2) (a) No — can multiply a scalar by an integer and yield a non-integer.

(b) Yes. $\rightarrow \infty$

(c) Yes. $\rightarrow \infty$

(d) Yes. $\rightarrow 2$ (not 3 since one component is constrained)

(e) No — can add -1 to -1 and be outside set.

(3) Shankar 1.3.1

Initial vectors $|A\rangle \Leftrightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $|B\rangle \Leftrightarrow \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

$$|1\rangle = \frac{|A\rangle}{|A|} \Leftrightarrow \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$|2'\rangle = |B\rangle - |1\rangle\langle 1|B\rangle$$

$$= \begin{pmatrix} 2 \\ -6 \end{pmatrix} - \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -6 \end{pmatrix} - \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \cdot \left(\frac{6}{5} - \frac{24}{5} \right)$$

$$= \begin{pmatrix} 2 \\ -6 \end{pmatrix} - \begin{pmatrix} -54/25 \\ -72/25 \end{pmatrix}$$

$$= \begin{pmatrix} 104/25 \\ -78/25 \end{pmatrix}$$

$$|2\rangle = \frac{|2'\rangle}{|2'|} = \frac{|2'\rangle}{\sqrt{16{,}900/625}} = \frac{|2'\rangle}{130/25} = (|2'\rangle)\frac{5}{26}$$

$$= \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

... which is unique unless you count $(|1\rangle, -|2\rangle)$ as a different basis.

Gram-Schmidt Theorem

1.3.2 $|I\rangle = \begin{bmatrix}3\\0\\0\end{bmatrix} \quad |II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} \quad |III\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix}$

$$|1\rangle = \frac{1}{\sqrt{[3,0,0]\begin{bmatrix}3\\0\\0\end{bmatrix}}}\begin{bmatrix}3\\0\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}3\\0\\0\end{bmatrix} = \boxed{\begin{matrix}|1\rangle\\ \parallel \\ \begin{bmatrix}1\\0\\0\end{bmatrix}\end{matrix}}$$

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

$$= \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}[1,0,0]\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\2\end{bmatrix} - \cancel{\begin{bmatrix}1\\0\\0\end{bmatrix}(0)}^{\,0}$$

$$|2'\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix}.$$

$$|2\rangle = \frac{1}{\sqrt{[0,1,2]\begin{bmatrix}0\\1\\2\end{bmatrix}}}\begin{bmatrix}0\\1\\2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix} = \boxed{\begin{matrix}|2\rangle\\ \parallel \\ \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\end{matrix}}$$

$$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle$$

$$= \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}[1\ 0\ 0]\begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\left[0\ \ 1/\sqrt{5}\ \ 2/\sqrt{5}\right]\begin{bmatrix}0\\2\\5\end{bmatrix}$$

$$= \begin{bmatrix}0\\2\\5\end{bmatrix} - \cancel{\begin{bmatrix}1\\0\\0\end{bmatrix}(0)} - \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\left(\frac{2}{\sqrt{5}} + \frac{10}{\sqrt{5}}\right)$$

$$|3'\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\12/5\\24/5\end{bmatrix} = \begin{bmatrix}0\\-2/5\\1/5\end{bmatrix}$$

$$|3\rangle = \frac{1}{\sqrt{[0,-2/5,\ 1/5]\begin{bmatrix}0\\-2/5\\1/5\end{bmatrix}}}\begin{bmatrix}0\\-2/5\\1/5\end{bmatrix} = \frac{1}{\left(\frac{1}{25}+\frac{1}{25}\right)^{\frac{1}{2}}}\begin{bmatrix}0\\-2/5\\1/5\end{bmatrix} = \begin{bmatrix}0\\-2\sqrt{5}/5\\\sqrt{5}/5\end{bmatrix} = \boxed{\begin{matrix}|3\rangle\\ \parallel \\ \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}\end{matrix}}$$

1.6.2) Given Ω and Λ are Hermitian what can we say about...

1) $\Omega\Lambda$? We know $\Omega = \Omega^\dagger$, $\Lambda = \Lambda^\dagger$

So, $\Omega\Lambda = \Omega^\dagger\Lambda^\dagger = (\Lambda\Omega)^\dagger$

$\Rightarrow$ If $\Omega\Lambda = \Lambda\Omega$ then $\Omega\Lambda$ is Hermitian

$\Rightarrow$ $\boxed{\text{If } [\Omega,\Lambda] = 0 \text{ then } \Omega\Lambda \text{ is Hermitian.}}$

2) $\Omega\Lambda + \Lambda\Omega$:

$$\Omega\Lambda + \Lambda\Omega = \Omega^\dagger\Lambda^\dagger + \Lambda^\dagger\Omega^\dagger = (\Lambda\Omega)^\dagger + (\Omega\Lambda)^\dagger$$
$$= (\Omega\Lambda)^\dagger + (\Lambda\Omega)^\dagger$$
$$= (\Omega\Lambda + \Lambda\Omega)^\dagger$$

$\Rightarrow$ $\boxed{\Omega\Lambda + \Lambda\Omega \text{ is Hermitian}}$

3) $[\Omega,\Lambda]$:

$$[\Omega,\Lambda] = \Omega\Lambda - \Lambda\Omega = \Omega^\dagger\Lambda^\dagger - \Lambda^\dagger\Omega^\dagger = (\Lambda\Omega)^\dagger - (\Omega\Lambda)^\dagger$$

$[\Omega,\Lambda] = [\Lambda,\Omega]^\dagger = -[\Lambda,\Omega]$ $\Rightarrow$ $\boxed{[\Omega,\Lambda] \text{ is antiHermitian}}$

4) $i[\Omega,\Lambda]$:

$$i[\Omega,\Lambda] = \{i[\Omega,\Lambda]\}^{\dagger\dagger}$$
$$= \{i^*[\Omega,\Lambda]^\dagger\}^\dagger$$
$$= \{-i(-[\Omega,\Lambda])\}^\dagger \quad \text{from part 3)}$$

$$i[\Omega,\Lambda] = \{i[\Omega,\Lambda]\}^\dagger$$

so $\boxed{i[\Omega,\Lambda] \text{ is Hermitian.}}$

(6) Shankar 1.6.4

Property 2: $\det U = \det(U^T) = [\det(U^\dagger)]^*$

Property 3: $\det(U^\dagger U) = (\det U)(\det U^\dagger)$

For a unitary matrix, $U^\dagger U = \mathbb{1}$. $\det(\mathbb{1}) = 1$

so
$$\begin{aligned} 1 &= \det(U^\dagger U) \\ &= (\det U^\dagger)(\det U) \\ &= (\det U)^*(\det U) \end{aligned}$$
QED.

(7) Shankar 1.6.5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
– Orthogonal: obvious.

– Unitary: $R^\dagger \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$$R^\dagger R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$$ ✓

1.81 $\det(\Omega - \omega I) = 0$

$$\det\begin{bmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{bmatrix}$$

Characteristic equation:

$$(1-\omega)[(2-\omega)(4-\omega)] - \cancel{3(0)} + \cancel{1(0)} = 0$$

$\Rightarrow \omega = 1, 2, 4$ are the eigenvalues.

To find the eigenvectors: $\omega = 1$;

$$\begin{bmatrix} 1-1 & 3 & 1 \\ 0 & 2-1 & 0 \\ 0 & 1 & 4-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

I) $3x_2 + x_3 = 0$

II) $x_2 = 0$

III) $x_2 + 3x_3 = 0$

So, $x_2 = 0$

III) $\Rightarrow x_3 = 0$

we let $x_1 = 1 \Rightarrow |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now $\omega = 2$

$$\begin{bmatrix} 1-2 & 3 & 1 \\ 0 & 2-2 & 0 \\ 0 & 1 & 4-2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -x_1 + 3x_2 + x_3 = 0$ let $x_2 = 1$.

$x_2 + 2x_3 = 0$ $\Rightarrow x_3 = -\frac{1}{2}$, $x_1 = \frac{5}{2}$

$x_1 = 3 - \frac{1}{2} = \frac{5}{2}$ $\Rightarrow |2'\rangle = \begin{bmatrix} 5/2 \\ 1 \\ -1/2 \end{bmatrix}$

1.8.1 (cont.) And finally,

$$\begin{bmatrix} 1-4 & 3 & 1 \\ 0 & 2-4 & 0 \\ 0 & 1 & 4-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_1 + 3x_2 + x_3 = 0$$
$$-2x_2 = 0$$
$$x_2 = 0$$

$$-3x_1 + x_3 = 0$$
$$x_3 = 3x_1 \qquad \text{let } x_1 = 1, \; x_3 = 3$$

$$\Rightarrow |4'\rangle = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Normalizing, $\boxed{|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ (already normalized!)

$$|2\rangle = \frac{|2'\rangle}{|2'|} = \frac{2}{\sqrt{30}}\begin{bmatrix} 5/2 \\ 1 \\ -1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 5/\sqrt{30} \\ 2/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix} \longleftarrow |2\rangle}$$

$$|4\rangle = \frac{|4'\rangle}{|4'|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{10} \\ 0 \\ 3/\sqrt{10} \end{bmatrix} \longleftarrow |4\rangle}$$

The matrix is not Hermitian. $\Omega^\dagger \neq \Omega$
The eigenvectors are not orthogonal

(9) Shankar 1.8.7

$$\det \Omega = \omega_1 \omega_2 = 1 - 4 = -3$$
$$\operatorname{Tr} \Omega = \omega_1 + \omega_2 = 1 + 1 = 2$$

So eigenvalues are presumably 3, −1.

Verifying: $\det \begin{pmatrix} 1-\omega & 2 \\ 2 & 1-\omega \end{pmatrix} = 0$

$$(1-\omega)^2 - 4 = 0$$
$$1 - \omega = \pm 2$$
$$\omega = 3, -1. \quad \checkmark$$

1.8.9) We have a collection of masses m_α located at $\vec{r}_\alpha$ and rotating with angular velocity around a common axis.

The total angular momentum is

$$\vec{l} = \sum_\alpha m_\alpha(\vec{r}_\alpha \times \vec{v}_\alpha), \text{ where } v_\alpha = \vec{\omega} \times \vec{r}_\alpha. \text{ Note } \vec{r}_\alpha = (r_{\alpha x}, r_{\alpha y}, r_{\alpha z}$$

Recall, $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Substituting, $\vec{l} = \sum_\alpha m_\alpha(\vec{r}_\alpha \times \vec{\omega} \times \vec{r}_\alpha)$, and applying the identity,

$$\vec{l} = \sum_\alpha m_\alpha\left[(\vec{\omega})(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha(\vec{r}_\alpha \cdot \vec{\omega})\right]$$

Lets write this in Dirac notation...

$$\vec{l} = \sum_\alpha m_\alpha\Big[\underbrace{|\omega\rangle\langle r_\alpha|r_\alpha\rangle}_{I} - \underbrace{|r_\alpha\rangle\langle r_\alpha|\omega\rangle}_{II}\Big]$$

Looking at the terms seperately:

$$I: |\omega\rangle\langle r_\alpha|r_\alpha\rangle = r_\alpha^2|\omega\rangle = r_\alpha^2 \underbrace{I}_{\text{identity operator } - I}|\omega\rangle = \begin{bmatrix} r_\alpha^2 & 0 & 0 \\ 0 & r_\alpha^2 & 0 \\ 0 & 0 & r_\alpha^2 \end{bmatrix}|\omega\rangle$$

$$II: \underbrace{|r_\alpha\rangle\langle r_\alpha|}_{\text{a } 3\times3 \text{ matrix}}\omega\rangle = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}\begin{bmatrix} r_x & r_y & r_z \end{bmatrix}|\omega\rangle = \begin{bmatrix} r_x r_x & r_x r_y & r_x r_z \\ r_y r_x & r_y r_y & r_y r_z \\ r_z r_x & r_z r_y & r_z r_z \end{bmatrix}|\omega\rangle$$

note: I've dropped the α-subscript for clarity.

⟶ ℓ

$$\vec{l} = \underbrace{\sum_\alpha m_\alpha \begin{bmatrix} r^2 - r_x r_x & -r_x r_y & -r_x r_z \\ -r_y r_x & r^2 - r_y r_y & -r_y r_z \\ -r_z r_x & -r_z r_y & r^2 - r_z r_z \end{bmatrix}}_{M} |\omega\rangle$$

If we adjust the notation $x \to 1, y \to 2, z \to 3$ and put the α's back we have:

$$M_{ij} = \sum_\alpha m_\alpha \left[r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j \right]$$

or more compactly

$$|l\rangle = M|\omega\rangle$$

1) Will $|l\rangle$ and $|\omega\rangle$ always be parallel?
$|l\rangle$ and $|\omega\rangle$ are parallel if they vary by a scalar multiple: $|l\rangle = \lambda|\omega\rangle$
Then, $M|\omega\rangle = \lambda|\omega\rangle$ so all $|\omega\rangle$ must be eigenvectors of M. But, M is 3×3 and Hermitian so it will have at most 3 eigenvectors. Picking a $|\omega\rangle$ that is not one of these 3 will not be parallel to $|l\rangle$

2) M is hermitian ($M^\dagger = M$) because it is symetric about its diagonal. Since all its elements are obviously rea[l] it is sufficient to observe that $M^T = M$.

3) See 1). We find the $|\omega\rangle$'s which are parallel to $|l\rangle$ by finding the eigenvalues and their corresponding eigenvectors.

4) For a sphere, the moment of inertia matrix, M_{sphere} is diagonal and the eigenvalues are the same, mr^2.