P4410 Problem set 12 solutions

Shankar 19.3.1

From Shankar Eq. 19.3.11

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 g^2}{\hbar^4[\mu_0^2 + 4k^2\sin^2(\theta/2)]^2}$$

$$\int d\sigma = \int \frac{4\mu^2 g^2}{\hbar^4[\mu_0^2 + 4k^2\sin^2(\theta/2)]^2}\, d\Omega. \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

$$\sigma = \int_0^{\pi}\int_0^{2\pi} \frac{4\mu^2 g^2 \sin\theta}{\hbar^4[\mu_0^2 + 4k^2\sin^2(\theta/2)]^2}\, d\phi\, d\theta$$

$$= \frac{8\pi\mu^2 g^2}{\hbar^4}\int_0^{\pi} \frac{\sin\theta}{[\mu_0^2 + 4k^2\sin^2(\theta/2)]^2}\, d\theta$$

Integration yields

$$= \frac{8\pi\mu_0^2 g^2}{\hbar^4}\left(\frac{2}{4k^2\mu_0^2 + \mu_0^4}\right)$$

$$= \frac{16\pi^2 g^2}{\hbar^4\mu_0^2}\left(\frac{1}{\frac{4k^2}{\mu_0^2} + 1}\right)$$ substituting $r_0 = 1/\mu_0$ and noting $r_0\mu_0 = 1$

$$\sigma = \frac{16\pi^2 g^2 r_0^2}{\hbar^4}\left(\frac{1}{4k^2 r_0^2 + 1}\right) = 16\pi^2 r_0^2\left(\frac{g\mu r_0}{\hbar^2}\right)^2 \frac{1}{1 + 4k^2 r_0^2}$$ ✓

Shankar 19.3.2

(1) $$f(\theta) = \frac{-2\mu}{\hbar^2}\int \frac{\sin qr'}{q} V(r')r'\,dr' \,, \quad V(r) = -V_0\underbrace{\Theta(r_0-r)}_{\text{localized to sphere of radius } r_0}$$

$$= \frac{+2\mu V_0}{\hbar^2 q}\int_0^{r_0} r\sin qr'\,dr'$$

Integration yields

$$= \frac{2\mu V_0}{\hbar^2 q}\left[-\frac{r_0\cos qr_0}{q} + \frac{\sin qr_0}{q^2}\right]$$

$$f(\theta) = \frac{2\mu V_0}{\hbar^2 q}\left[\frac{\sin(qr_0) - r_0 q\cos(qr_0)}{q^2}\right]$$

and $$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\mu V_0}{\hbar^2 q^3}\right)^2 \left(\sin(qr_0) - r_0 q\cos(qr_0)\right)^2 \left(\frac{r_0}{r_0}\right)^6$$

$$\frac{d\sigma}{d\Omega} = 4r_0^2\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2\left[\frac{\left(\sin(qr_0) - qr_0\cos(qr_0)\right)^2}{(qr_0)^6}\right]$$

Shankar 19.3.2

(2) As $kr_0 \to 0$, $qr_0 \to 0$

We examine the factor in brackets from part (1)

$$\lim_{\epsilon\to 0} \frac{(\sin\epsilon - \epsilon\cos\epsilon)^2}{\epsilon^6}.$$ Taylor expand.

$$\lim_{\epsilon\to 0} \frac{\left(\epsilon - \frac{\epsilon^3}{3!} + \dots - \epsilon\left(1 - \frac{\epsilon^2}{2!} + \dots\right)\right)^2}{\epsilon^6} = \lim_{\epsilon\to 0} \frac{\left(\cancel{\epsilon} - \frac{\epsilon^3}{3!} + \dots - \cancel{\epsilon} + \frac{\epsilon^3}{2!} + \dots\right)^2}{\epsilon^6}$$

$$= \lim_{\epsilon\to 0} \frac{\left(\epsilon^3\left(\frac{1}{2} - \frac{1}{6}\right)\right)^2}{\epsilon^6} = \lim_{\epsilon\to 0} \frac{\epsilon^6}{\epsilon^6}\frac{1}{9}.$$ Clearly (or applying l'Hopital's rule 6 times) this yields $\frac{1}{9}$.

Thus picking up 4π from angle independent $\int d\Omega$ we obtain

$$\sigma = \frac{16\pi r_0^2}{9}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2$$ ✓

Shankar 19.5.1 100 MeV neutron incident on nucleus

$l_{max} \cong k r_0$ $\qquad r_0 \cong 1F = 10^{-5} Å$

$k = \frac{\sqrt{2EM}}{\hbar}$ $\qquad E = 100$ MeV, $M = 938$ MeV
$\hbar c \cong 200$ MeV F

$$l_{max} \cong \frac{\sqrt{2 \cdot 100 MeV \cdot 938 MeV}}{200 MeV \cancel{F}} \cancel{1F}$$

$\cong 2.17$

or $l_{max} \cong 2$ ✓

④ Shankar 19.5.3

1) $\lim_{kr_0 \to 0} \delta_l = (kr_0)^{2l+1}$

For small kr_0, only need to consider $l=0$

so $\delta_0 \approx kr_0$

$\sigma \approx \sigma_0 \approx \frac{4\pi}{k^2} \sin^2\delta_0 \approx \frac{4\pi \delta_0^2}{k^2} \approx 4\pi r_0^2$

Note that this is 4x larger than classical geometric cross-section.

2) $\delta_l = \arctan\left[\frac{j_l(kr_0)}{n_l(kr_0)}\right] \underset{kr_0 \to \infty}{\approx} \arctan\left(\frac{\sin\left(kr_0 - l\frac{\pi}{2}\right)/\cancel{kr_0}}{-\cos\left(kr_0 - l\frac{\pi}{2}\right)/\cancel{kr_0}}\right)$

$$= -\left(kr_0 - l\frac{\pi}{2}\right)$$

so $\sin^2\delta_l \longrightarrow \sin^2\left(kr_0 - l\frac{\pi}{2}\right)$ QED.

(5) For $\ell = 0$, wave functions are spherically symmetric: $\psi = R(r)$. Use radial equation (Shankar 12.6.5):

$$\left\{ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]\right\} U(r) = 0$$

where $U(r) = rR(r)$, and $\ell = 0$:

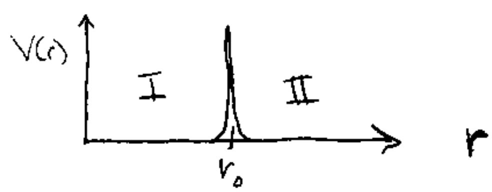
$$\frac{d^2 U}{dr^2} + \left(\frac{2mE}{\hbar^2} - \frac{2mV(r)}{\hbar^2}\right) U(r) = 0$$

now $E = \frac{\hbar^2 k^2}{2m}$ so

$$\frac{d^2 U}{dr^2} + \left(k^2 - \frac{2mV(r)}{\hbar^2}\right) U(r) = 0$$

Solutions in $V=0$ regions:
$U(r) = Ae^{ikr} + Be^{-ikr}$

Divide up regions radially:



Region I: $U_I(r) = A'e^{ikr} + B'e^{-ikr}$ constraint $A' = -B'$ due to $U(0) = 0$.
so $U_I = A \sin kr$, redefining $A = \frac{2A'}{i}$

$$U_{II}(r) = Ce^{ikr} + De^{-ikr}$$

Constraints on $U_{II}(r)$:

$U_{II}(r_0) = U_I(r_0)$: $Ce^{ikr_0} + De^{-ikr_0} = A \sin kr_0$ (*)

Constraints on $\frac{dU}{dr}$ at r_0:

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} dr \left\{ \frac{d^2 U}{dr^2} + k^2 U(r) - \frac{2ma\delta(r-r_0)}{\hbar^2} U(r) \right\} = 0$$

As on 1-D Schrödinger eqn. problems, take $\varepsilon \to 0$:

$$\left.\frac{dU_{II}}{dr}\right|_{r_0} - \left.\frac{dU_I}{dr}\right|_{r_0} - \underbrace{\varepsilon k^2 U(r_0)}_{0} - 2\frac{ma}{\hbar^2}U(r_0) = 0.$$

$$iCke^{ikr_0} - iDke^{-ikr_0} - Ak\cos kr_0 = \frac{2ma}{\hbar^2} A \sin kr_0$$

Substitute $A \to \dfrac{Ce^{ikr_0} + De^{-ikr_0}}{\sin kr_0}$ from $(*)$.

$$iCke^{ikr_0} - iDke^{-ikr_0} - k\left(Ce^{ikr_0} + De^{-ikr_0}\right)\cot kr_0 = \frac{2ma}{\hbar^2}\left(Ce^{ikr_0} + De^{-ikr_0}\right)$$

$$C\left[ike^{ikr_0} - ke^{ikr_0}\cot kr_0 - \frac{2ma}{\hbar^2}e^{ikr_0}\right] = D\left[\frac{2ma}{\hbar^2}e^{-ikr_0} + ike^{-ikr_0} + ke^{-ikr_0}\cot kr_0\right]$$

$$\frac{C}{D} = e^{-2ikr_0}\,\frac{ik + k\cot kr_0 + \frac{2ma}{\hbar^2}}{ik - k\cot kr_0 - \frac{2ma}{\hbar^2}}$$

$$= e^{-2ikr_0}\left[\frac{e^{ikr_0} + \frac{2ma}{\hbar^2 k}}{-e^{-ikr_0} - \frac{2ma}{\hbar^2 k}}\right] = -e^{2i\delta_0}$$

so
$$e^{2i\delta_0} = e^{-2ikr_0}\left[\frac{e^{ikr_0} + \frac{2ma}{\hbar^2 k}}{e^{-ikr_0} + \frac{2ma}{\hbar^2 k}}\right]$$

$$\frac{d\sigma_0}{d\Omega} = |f_0(\theta)|^2 = \left|\frac{1}{k}e^{i\delta_0}\sin\delta_0\right|^2 = \frac{1}{k^2}\sin^2\delta_0$$

$$2i\sin 2\delta_0 = e^{-2ikr_0}\left[\frac{e^{ikr_0} + \frac{2ma}{\hbar^2 k}}{e^{-ikr_0} + \frac{2ma}{\hbar^2 k}}\right] - e^{2ikr_0}\left[\frac{e^{-ikr_0} + \frac{2ma}{\hbar^2 k}}{e^{ikr_0} + \frac{2ma}{\hbar^2 k}}\right]$$

Define: $x \equiv kr_0$; $\beta \equiv \dfrac{2ma}{\hbar^2 k}$

$$2i\sin 2\delta_0 = \frac{e^{-ix} + \beta e^{-2ix}}{e^{-ix}+\beta} - \frac{e^{ix} + \beta e^{2ix}}{e^{ix}+\beta}$$

$$= \frac{\left(e^{-ix}+\beta e^{-2ix}\right)\left(e^{ix}+\beta\right) - \left(e^{ix}+\beta e^{2ix}\right)\left(e^{-ix}+\beta\right)}{\left(e^{-ix}+\beta\right)\left(e^{ix}+\beta\right)}$$

$$= \frac{\left(1+2\beta e^{-ix}+\beta^2 e^{-2ix}\right) - \left(1+2\beta e^{ix}+\beta e^{2ix}\right)}{1+2\beta\cos x+\beta^2}$$

$$= \frac{-4i\beta\sin x - 2\beta^2 i\sin 2x}{1+2\beta\cos x+\beta^2} \qquad \text{Take } x \ll 1 \text{ approx:}$$

$$2i\sin 2\delta_0 \cong \frac{-4ix\left(\beta-\beta^2\right)}{(1+\beta)^2} = -4ix\frac{\beta}{1+\beta}$$

$$\sin 2\delta_0 \approx -2kr_0 \frac{\frac{2ma}{\hbar^2 k}}{1+\frac{2ma}{\hbar^2 k}}$$

$$\text{so} \quad \sin\delta_0 \approx -kr_0 \frac{\frac{2ma}{\hbar^2 k}}{1+\frac{2ma}{\hbar^2 k}}$$

$$\frac{d\sigma}{d\Omega} = r_0^2\left(\frac{2ma/\hbar^2 k}{1+\frac{2ma}{\hbar^2 k}}\right)^2$$

since spherically symmetric, $\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 4\pi r_0^2\left(\frac{\frac{2ma}{\hbar^2 k}}{1+\frac{2ma}{\hbar^2 k}}\right)^2$

Note that this has correct limiting behavior:

$a\to 0$: $\sigma\to 0$ since there's no potential

$a\to\infty$: looks like hard sphere since the "shell" becomes impenetrable: $\sigma \to 4\pi r_0^2$.