(1) Shankar 18.2.2

$$\vec{E}(t) = \mathcal{E} e^{-t^2/\tau^2}\, \vec{e}_z \qquad \text{so } V = -e\mathcal{E}\underbrace{\hat{r}\cos\hat{\theta}}_{=z} e^{-t^2/\tau^2}$$

$$C_{n=1,0,0}^{n=2,l,m} = \frac{-i}{\hbar}\int_{-\infty}^{\infty} dt\, e^{i\omega t} e^{-t^2/\tau^2}(-e\mathcal{E}) \langle 2lm|\hat{r}\cos\hat{\theta}|100\rangle$$

recall that $\langle\vec{r}|1\,0\,0\rangle$ has no θ,ϕ dependence, and $r\cos\theta = \sqrt{\frac{3}{4\pi}}\,Y_{10}(\theta,\phi)\cdot r$ so $\langle 2lm|\hat{r}\cos\hat{\theta}|100\rangle \neq 0$ only for $l=1$, $m=0$ since everything else is orthogonal to $\hat{r}\cos\hat{\theta}|100\rangle$.

$$\langle 210|\hat{r}\cos\theta|100\rangle = \int_0^\infty dr\, r^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\, R_{21}(r) Y_{10}^*(\theta)\, r\cos\theta\, R_{10}(r) Y_{00}$$

$$= \frac{\sqrt{3}}{4\pi}\cdot 2\pi \cdot \int_{-1}^{1} dx\, x^2 \int_0^\infty dr\, r^3 R_{21}(r) R_{10}(r) \qquad \text{where } x \leftrightarrow \cos\theta$$

$$= \frac{1}{\sqrt{3}}\int_0^\infty dr\, r^3 R_{21}(r) R_{10}(r) = \frac{1}{\sqrt{3}}\int_0^\infty dr\, r^2 \frac{1}{\sqrt{24}}\frac{1}{a_0^{5/2}}\, r e^{-r/2a_0}\, r \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

$$= \frac{1}{\sqrt{12}}\int_0^\infty dr\, r^4 e^{\frac{-3r}{2a_0}} \frac{1}{a_0^4} = \frac{1}{\sqrt{12}}\frac{4!}{a_0^4}\left(\frac{2a_0}{3}\right)^5 = \sqrt{2}\,\frac{2^7}{3^5} a_0$$

$$\text{So } C_{100}^{210} = \frac{-i}{\hbar}\sqrt{2}\,\frac{2^7}{3^5} a_0 (-e\mathcal{E}) \underbrace{\int_{-\infty}^{\infty} dt\, e^{i\omega t} e^{-t^2/\tau^2}}_{=\sqrt{\pi}\,\tau e^{-\left(\frac{\omega\tau}{2}\right)^2}}$$

$$C_{100}^{210} = \frac{-i}{\hbar} \frac{2^{15/2}}{3^5} a_0 (-e\mathcal{E}) \sqrt{\pi}\, \tau\, e^{-\left(\frac{\omega t}{2}\right)^2}$$

$$P(n=1 \to n=2) = \left| C_{100}^{210} \right|^2 = \frac{e^2 \mathcal{E}^2 a_0^2}{\hbar^2} \frac{2^{15}}{3^{10}} \pi \tau^2 e^{-\frac{\omega^2 \tau^2}{2}}$$

2a) Start w/

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t_f} \langle f^0 | H'(t') | i^0 \rangle e^{i\omega_{fi}t'} dt'$$

where $H'(t') = V'(t) = \alpha\,\delta(x - ct')$

so $\langle f^0 | V'(t') | i^0 \rangle = \int dx\, dx' \langle f^0 | x \rangle \langle x | V'(t') | x' \rangle \langle x' | i^0 \rangle$

$$= \alpha \int dx\,dx' \langle f^0 | x \rangle \delta(x - ct')\,\delta(x - x') \langle x' | i^0 \rangle$$

$$= \alpha \int_{-\infty}^{\infty} dx\, \psi_{f^0}^*(x)\,\delta(x - ct')\,\psi_{i^0}(x)$$

$$= \alpha\, \psi_{f^0}^*(ct')\,\psi_{i^0}(ct')$$

So we have

$$\boxed{d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{\infty} \alpha\, \psi_{f^0}^*(ct')\,\psi_{i^0}(ct')\, e^{i\omega_{fi}t'} dt'}$$

And probability is $\boxed{|d_f(t)|^2}$

(b) $V(x,t) = \frac{a}{2\pi c}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\left(\frac{x}{c}-t\right)}$

$d_f(t)$ is now $\frac{-ia}{\hbar 2\pi c}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \quad \psi_f^*(x)\, e^{i\omega\left(\frac{x}{c}-t\right)}\, e^{i\omega_{fi}t}\, \psi_i(x)$

Rearrange integration:

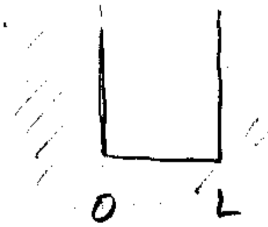
$$\frac{-ia}{2\pi\hbar c}\int d\omega \int dx\, e^{i\omega\frac{x'}{c}}\, \psi_f^*(x)\psi_i(x) \underbrace{\int dt\, e^{i(\omega_{fi}t-\omega t)}}_{=2\pi\delta(\omega_{fi}-\omega)}$$

$$= \frac{-ia}{\hbar c}\int dx\, e^{i\omega_{fi}x/c}\, \psi_f^*(x)\psi_i(x)$$

Change variables $x \to ct'$ $\quad dx = c\,dt'$

$$= \frac{-ia}{\hbar}\int dt'\, e^{i\omega_{fi}t'}\, \psi_f^*(ct')\psi_i(ct')$$ which is same as part a).

c)



$$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{otherwise} \end{cases}$$

Solutions are
$$\psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) ; & 0<x<L \\ 0 & \text{otherwise} \end{cases}$$

Energies $\Rightarrow$ $E = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$; $n = 1, 2, \cdots$

Ground state $\psi_0(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

1st excited $\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$

$$\omega_{10} = \frac{E_1 - E_0}{\hbar} = \frac{3\pi^2\hbar}{2mL^2}$$

So
$$\langle f^\circ | V'(t') | i^\circ \rangle = \int_0^L \psi_1^*(x)\, \alpha\, \delta(x-ct')\, \psi_0(x)\, dx$$
$$= \alpha \int_0^L \frac{2}{L} \sin\left(\frac{2\pi x}{L}\right) \delta(x-ct') \sin\left(\frac{\pi x}{L}\right) dx$$
$$= \alpha \frac{2}{L} \sin\left(\frac{2\pi}{L} ct'\right) \sin\left(\frac{\pi c t'}{L}\right) \quad ; \quad 0 < t' < L/c$$

And
$$d_{10}(t) = -\frac{i}{\hbar} \int_0^{L/c} \alpha \frac{2}{L} \sin\left(\frac{2\pi}{L} ct'\right) \sin\left(\frac{\pi c t'}{L}\right) \exp\left(\frac{i3\pi^2\hbar}{2mL^2} t'\right) dt'$$

Let $u = \frac{\pi c t'}{L}$ $\Longrightarrow$ $-\frac{i}{\hbar} \frac{2\alpha}{L} \frac{L}{\pi c} \int_0^\pi \sin(2u) \sin(u) \exp(i\beta u)\, du$; $\beta = \frac{3}{2}\frac{\pi\hbar}{mcL}$

$$-\frac{i2\alpha}{\pi c\hbar}\left[-\frac{4i(1+e^{i\pi\beta})\beta}{9-10\beta^2+\beta^4}\right] = -\frac{8}{\pi c\hbar}\left[\frac{i(1+e^{i\pi\beta})\beta}{9-10\beta^2+\beta^4}\right]$$

$$\boxed{\text{Prob} = \left| -\frac{8}{\pi c\hbar}\left[\frac{i(1+e^{i\pi\beta})}{9-10\beta^2+\beta^4}\right] \right|^2}$$