P4410 | LECTURE 9

Continuing with direct product spaces: Basis representations.

Possible bases for 2 particles (or 1 in 2 dimensions):

$\hat{X}_1, \hat{X}_2$ where $\hat{X}_1$ is particle 1 momentum:

$$\langle x_1 x_2 | \psi \rangle = \psi(x_1, x_2)$$

← You can think of $\hat{X}_1$ being degenerate, with degeneracy broken by $\hat{X}_2$.

$\hat{P}_1, \hat{P}_2$: $\langle p_1 p_2 | \psi \rangle = \psi(p_1, p_2)$

or even $\hat{X}_1, \hat{P}_2$: $\langle x_1 p_2 | \psi \rangle = \psi(x_1, p_2)$ ← Note: $[\hat{X}_1, \hat{P}_2] = 0$, so we can form a basis of simultaneous eigenstates.

(What are operators $\hat{X}_{1,2}, \hat{P}_{1,2}$ in these bases?)

Time evolution: follows the Schrödinger eqn with $\hat{H}$ defined in the $\mathbb{V}_{1\otimes 2}$ space: $i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$.

Special (but common) case: separable Hamiltonians:

$$\hat{H} = \hat{H}_1(\hat{X}_1, \hat{P}_1) + \hat{H}_2(\hat{X}_2, \hat{P}_2).$$

← i.e. particles/dimensions are independent.

A stationary state $|\psi(t)\rangle = e^{-iEt/\hbar}|\psi(0)\rangle$ and since $[\hat{H}_1, \hat{H}_2] = 0$ we can pick a basis where both $\hat{H}_1, \hat{H}_2$ diagonal so $E = E_1 + E_2$:

$$|\psi(t)\rangle = e^{-i(E_1+E_2)t/\hbar}|\psi(0)\rangle = e^{\frac{-iE_1 t}{\hbar}} e^{\frac{-iE_2 t}{\hbar}}|\psi(0)\rangle.$$

If independent particles have wave functions $\psi_1(x), \psi_2(x)$, then wave function in combined space must obey

$\langle x_1 x_2 | E \rangle = \psi_E(x_1, x_2)$ <u>and</u>

$$\hat{X}_1|x_1 x_2\rangle = x_1|x_1 x_2\rangle, \quad \hat{X}_2|x_1 x_2\rangle = x_2|x_1 x_2\rangle$$

Assume a separable solution: $|\psi(t)\rangle = \left[|E_1\rangle e^{\frac{-iE_1 t}{\hbar}}\right] \otimes \left[|E_2\rangle e^{\frac{-iE_2 t}{\hbar}}\right]$

In x_1, x_2 basis:

$$\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1) - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi_E(x_1,x_2) = (E_1+E_2)\psi_E(x_1,x_2)$$

Assume $\psi_E = \psi_{E_1}(x_1)\psi_{E_2}(x_2)$

$$\underbrace{\frac{\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1)\right]\psi_{E_1}(x_1)}{\psi_{E_1}(x_1)}}_{\text{Fn. of } x_1 \text{ alone}} + \underbrace{\frac{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi_{E_2}(x_2)}{\psi_{E_2}(x_2)}}_{\text{Fn. of } x_2 \text{ alone}} = \underbrace{E_1+E_2}_{\text{constant}}$$

→ Each function must be constant individually. So we have 2 equations that each look like 1-D Schröd. eqn.

For separable Hamiltonian, this generalizes to N dimensions/particles.

Identical particles:

Recall from last semester.

Identical particles are completely indistinguishable, so any state that has different physical properties for any two of them can't exist. So state must obey:

$$\hat{P}_{12}|\psi\rangle = e^{i\delta}|\psi\rangle$$ where δ is real, $\hat{P}_{12}$ switches "2" and "1".

Note that this is unique to the multi-particle case, not applicable to 1 particle in N dimensions.

But $\hat{P}_{12}^2 = 1$, so $e^{2i\delta} = 1$. $\therefore \delta = 0$ or π

and $\hat{P}_{12}|\psi\rangle = \pm|\psi\rangle$.

Fermions $\longrightarrow$ always $-$

Bosons $\longrightarrow$ always $+$

so $|x_1 x_2\rangle$ is not a legitimate state if $x_1 \neq x_2$ and particles are identical, because it associates position x_1 with particle 1 and x_2 with particle 2. If we want a state with one particle at x_1 and another at x_2, then we must symmetrize (bosons) or antisymmetrize (fermions) the state:

$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|x_1 x_2\rangle \pm |x_2 x_1\rangle\Big)$. If you have N particles, must add <u>all</u> permutations (i.e. N!) with a + sign (bosons) or (fermions) a + sign for even # of flips and a $-$ for odd # of flips: $\sum_{\text{permutations}} \varepsilon_{ijk\cdots}|\psi\rangle$

Special fermion case: Take two identical fermions in states $|\psi_1\rangle, |\psi_2\rangle$: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1 \psi_2\rangle$

Must antisymmetrize state:

$$|\psi\rangle = \Big(|\psi_1\psi_2\rangle - |\psi_2\psi_1\rangle\Big)/\sqrt{2}$$

If $|\psi_1\rangle = |\psi_2\rangle$, then $|\psi\rangle = 0$ $\rightarrow$ Pauli exclusion principle.

N identical fermions must be in <u>mutually</u> orthogonal states.

Identical particle Hilbert spaces.

The (anti)symmetrization requirement is really a definition of the Hilbert spaces allowed for bosons and fermions. The vectors obeying these symmetry properties form complete vector spaces:

- Any combination of (anti) symmetric vectors is (anti) symmetric.

Basis states: Sums over basis states must include only distinct states. i.e. if 2 identical bosons can be in 3 possible states $|\alpha\rangle$ $|\beta\rangle$ $|\gamma\rangle$, then the basis states are

$$\left.\begin{array}{l} |\alpha\alpha\rangle \\ |\alpha\beta\rangle + |\beta\alpha\rangle \\ |\alpha\gamma\rangle + |\gamma\alpha\rangle \\ |\beta\beta\rangle \\ |\beta\gamma\rangle + |\gamma\beta\rangle \\ |\gamma\gamma\rangle \end{array}\right\} \text{6 basis states} \rightarrow \text{Hilbert space dimension is 6}$$

Distinguishable $\rightarrow$ dimension 9

Indistinguishable fermions $\rightarrow$ dimension 3

Hamiltonians must also be symmetric:

Helium atom: 2 electrons (fermions) orbiting nucleus:

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + \frac{\hat{P}_2^2}{2m} + V_e(\vec{r}_1) + V_e(\vec{r}_2) + e\frac{1}{|\vec{r}_1 - \vec{r}_2|}$$