P4410 LECTURE 8

A word on commutators: —read p. 129-133.

$[\hat{\Omega}, \hat{\Lambda}] \equiv \hat{\Omega}\hat{\Lambda} - \hat{\Lambda}\hat{\Omega}$ which is an <u>operator</u>.

Compatible operators: $[\hat{\Omega}, \hat{\Lambda}] = 0$:
- doesn't matter what order you measure.
- Can form a basis where both diagonal
- (i.e. a complete basis of simultaneous eigenvectors)

OR = something with NO zero eigenvalue: (i.e. $[\hat{x}, \hat{p}] = i\hbar\mathbb{1}$)
- NO eigenvectors in common.

$\longrightarrow$ uncertainty relation

OR = something with zero & nonzero eigenvalues:

(i.e. $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$)
- Operators have <u>some</u> common eigenstates
- <u>Can't</u> form a complete basis from them

Free particle propagator, in 1 dimension.

Vector space dimension is <u>infinite</u>. $\hat{H} = \frac{\hat{p}^2}{2m}$

Convenient bases: $\hat{x}, \hat{p}$ (nondegenerate), $\hat{H}$ (degenerate).

Calculate propagator in $\hat{x}$ basis: $\langle x | \hat{U}(t) | x' \rangle$

$$\hat{U} = \int dp\, |p\rangle\langle p|\, e^{-ip^2 t/2m\hbar} \quad \left(\text{i.e. } \sum_E |E\rangle\langle E| e^{-iEt/\hbar}\right)$$

$$\text{so } U_{xx'}(t) = \int dp\, \langle x|p\rangle\langle p|x'\rangle\, e^{-ip^2 t/2m\hbar} = \int dp\, e^{\frac{ipx}{\hbar}} e^{\frac{-ipx'}{\hbar}} e^{-ip^2 t/2m\hbar}$$

$$= (\text{from table}) = \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{im(x-x')^2/2\hbar t}$$

So to solve an initial-state problem $\langle x|\psi(t)\rangle = \langle x|\hat{U}|\psi(0)\rangle$

$$\psi(x,t) = \int dx' \langle x|\hat{U}|x'\rangle\langle x'|\psi(0)\rangle$$

$$= \int dx'\, U_{xx'}(t)\,\psi(x',0)$$

Can treat matrix element $U_{xx'}(t)$ as the amplitude for a particle initially at x' to be found at x at time t. In another basis, substitute "basis state label" (i.e. $p, k, L_z \ldots$) for x. In the energy basis, $\hat{U}$ is diagonal — as expected since energy eigenstates are stationary: there is no probability for a particle in one E eigenstate at time 0 to be in a different one at time t. (If $\frac{\partial \hat{H}}{\partial t} \neq 0$, then $\hat{U}$ depends on t_0, t not $0, t$).

Gaussian wave packet: $\psi(x,t=0) = e^{\frac{ip_0 x}{\hbar}} \dfrac{e^{-x^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$

Last semester's way to solve:
- transform to k basis,
- Apply k-dependent phase factor, (i.e. $e^{-i\hbar k^2 t/2m}$)
- transform back to x basis.

But now that we have propagator in x basis, we can do the whole problem without a basis transformation:

$$\psi(x,t) = \int dx'\, U_{xx'}(t)\,\psi(x',0)$$

$$= \int dx' \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{im(x-x')^2}{2\hbar t}}\, e^{\frac{ip_0 x}{\hbar}}\, e^{-x^2/2\Delta^2}\, \frac{1}{(\pi\Delta^2)^{1/4}}$$

$$= \sqrt{\frac{m}{2\pi\hbar i t\sqrt{\pi}\,\Delta}} \int \exp\left(\frac{im(x-x')^2}{2\hbar t} + \frac{ip_0 x}{\hbar} - \frac{x^2}{2\Delta^2}\right) dx'$$

$$= \text{(from Shankar)} \quad \frac{1}{\sqrt{\sqrt{\pi}\left(\Delta + \frac{i\hbar t}{m\Delta}\right)}} \exp\left[\frac{-\left(x - \frac{p_0}{m}t\right)^2}{2\Delta^2\left(1 + i\hbar t/m\Delta^2\right)}\right] \exp\left[\frac{ip_0}{\hbar}\left(x - \frac{p_0 t}{2m}\right)\right]$$

which has expected features:

- peak of packet propagates at speed $\frac{p_0}{m}$
- packet spreads at rate which grows with $\frac{1}{\Delta^2}$
- $e^{i(kx-\omega t)}$ factor present too.

I'm assuming you remember well:

- particle in box
- harmonic oscillator (including energy basis, $\hat{a}^\dagger, \hat{a}$, etc)
- Uncertainty relation games
- Momentum basis

If not, go back to notes & examine chapt. 5, 7, 9.

QM with more degrees of freedom (higher dimensions):

If you add a second particle to a problem, you must expand the Hilbert space accordingly. If they are distinguishable particles, the dimension of the new space is product of dimensions of individual spaces:

particle 1 states: space $\mathbb{V}_1$ particle 2 states: space $\mathbb{V}_2$

$|1\rangle, |2\rangle, \ldots, |n\rangle$ $|A\rangle, |B\rangle, |C\rangle, \ldots, |m\rangle$

combined states:

$$\left.\begin{array}{l} |1A\rangle, |1B\rangle, |1C\rangle, \ldots, |1m\rangle \\ |2A\rangle, |2B\rangle, |2C\rangle, \ldots, |2m\rangle \\ |nA\rangle, |nB\rangle, |nC\rangle, \ldots, |nm\rangle \end{array}\right\} \text{combined space has dimension } nm.$$

Shankar calls this a direct product space, $\mathbb{V}_1 \otimes \mathbb{V}_2$.