P4410 LECTURE 7

Review of QM postulates & time evolution.

1. State of a system is described by a vector in Hilbert space at time t: $|\psi(t)\rangle$. It's a bad idea to think of this as standing for a spatial wave function $\psi(x,t)$, which is only the x-basis representation of the vector — and there's nothing special about $\langle x|\psi(t)\rangle$ vs. any other basis representation. What's "special" is only notational, since most authors use the ket label ψ as the variable name $\psi(x)$ — and some other letter for other continuous basis (i.e. $b(k)$ in Liboff) Shankar just calls it $\psi(k)$ which is also misleading since of course it's not the same function as $\psi(x)$.

2. Observables (such as x, p) are represented by Hermitian operators. In particular, the matrix element of a generalized coordinate operator $\hat{x}$ in its basis is $\langle x|\hat{x}|x'\rangle = x\,\delta(x-x')$, and the matrix element of its conjugate momentum is:
$\langle x|\hat{p}|x'\rangle = -i\hbar\,\delta'(x-x')$.

3. An ideal measurement of an observable represented by operator $\hat{\Omega}$ will yield an eigenvalue ω, with probability $|\langle\omega|\psi\rangle|^2$. After the measurement the system is left in state $|\omega\rangle$. Any info about $|\psi\rangle$ before is lost.
Exception: If ω is degenerate, then the system is left in some vector in the degenerate subspace $|\omega\rangle$, preserving as much info about $|\psi\rangle$ consistent with that.

④ The State evolves in time according to the Schrödinger equation: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$, where the form of $\hat{H}$ depends on the details of the system. In the case where the state is an eigenvector of $\hat{H}$ $\left(\hat{H}|\psi(t)\rangle = E|\psi(t)\rangle\right)$ the time evolution is simple: $|\psi(t)\rangle = e^{-\frac{iEt}{\hbar}}|\psi(0)\rangle$. This is a <u>stationary</u> state. Only changes by phase → all expectation values constant.

Generalizing to non-stationary states: solution is always expressible in form $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$. Finding matrix elements of $\hat{U}$ effectively solves the problem. In general:

$$i\hbar \frac{d}{dt}\left(\hat{U}(t)|\psi(0)\rangle\right) = \hat{H}\left(\hat{U}(t)|\psi(0)\rangle\right) \Rightarrow \dot{\hat{U}} = \frac{-i}{\hbar}\hat{H}\hat{U}$$

satisfied if $\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}}$. The operator can be <u>defined</u> using a Taylor series expansion in $\hat{H}$, but that's rarely useful for solving problems. Much more useful is to transform to the energy basis, where $\hat{H}$ is diagonal: $\hat{H}|i\rangle = E_i|i\rangle \Rightarrow e^{-i\hat{H}t/\hbar}|i\rangle = e^{-iE_it/\hbar}|i\rangle$.

Assume we know orthonormal eigenstates of $\hat{H}$: $\hat{H}|i\rangle = E_i|i\rangle$

$$\text{Then } \hat{U}(t) = \sum_i |i\rangle\langle i|\hat{U} = \sum_{i,j}|i\rangle\langle i|\hat{U}|j\rangle\langle j|$$

$$= \sum_{i,j}|i\rangle e^{-iE_jt/\hbar}\delta_{ij}\langle j|$$

$$\hat{U} = \sum_i e^{-iE_it/\hbar}|i\rangle\langle i|$$

Degenerate case: $\sum_E \sum_\alpha e^{-iEt/\hbar}|E,\alpha\rangle\langle E,\alpha|$

Time evolution is easy in energy basis.

Quick proof: expectation values don't change in stationary state. Let $|\psi(0)\rangle = |i\rangle$.

$$\langle\psi(t)|\hat{\Omega}|\psi(t)\rangle = \langle\psi(0)|\hat{U}^\dagger \hat{\Omega}\hat{U}|\psi(0)\rangle$$

$$= \langle i|\hat{U}^\dagger \hat{\Omega}\hat{U}|i\rangle$$

$$= \langle i|e^{iE_i t/\hbar}\hat{\Omega}e^{-iE_i t/\hbar}|i\rangle$$

$$= \langle i|\hat{\Omega}|i\rangle = \langle\Omega\rangle_{t=0}.$$

Propagator for a time-dependent Hamiltonian
(will be important for time-dependent perturbation theory)

$\hat{H} = \hat{H}_0 + \hat{H}_1(t)$ i.e. $\hat{H}_0$ is time-independent, large
$\hat{H}_1$ is time-dependent, small

Now $\hat{U}$ has to depend on <u>two</u> times: $\hat{U}(t_2,t_1)|\psi(t_1)\rangle = |\psi(t_2)\rangle$.

Generally $\hat{U}(t,0) = \mathbb{1} - \frac{i}{\hbar}\int_0^t dt'\, \hat{H}(t')\hat{U}(t')$

→ can be very complicated!

Generally:

$\hat{U}(t_2,t_1)$ is still unitary

$\hat{U}(t_2,t_1) = \hat{U}^\dagger(t_1,t_2)$ → can go "backwards"

$\hat{U}(t_3,t_2)\hat{U}(t_2,t_1) = \hat{U}(t_3,t_1)$