P4410 LECTURE 6

$\delta'(x-x')$ has integration property

$$\int dx'\, \delta'(x-x') f(x') = \left.\frac{df}{dx'}\right|_x .$$

So we can write $\delta'(x-x') = \delta(x-x')\frac{d}{dx}$.

Is $\hat{D}$ Hermitian?

$$D^*_{xx'} \stackrel{?}{=} D_{x'x}$$

No: $\delta'(x'-x) = -\delta'(x-x') \Leftarrow$ $\delta(x)$ is even, so $\delta'(x)$ is odd.

But can make it Hermitian by multiplying $-i$: $\hat{K} = -i\hat{D}$

$$K_{xx'} = -i\delta'(x-x') \qquad K_{x'x} = -i\delta'(x'-x) = i\delta'(x-x') = K^*_{xx'}$$

OK — but this doesn't rigorously show Hermiticity, it turns out, because the δ-functions are only meaningful when integrated:

Need $\langle G|\hat{K}|F\rangle = \langle F|\hat{K}|G\rangle^*$ for any vectors F, G.

Let $\langle x|F\rangle = f(x)$, $\langle x|G\rangle = g(x)$. Insert $1 = \int dx\, |x\rangle\langle x|$:

$$\langle G|\hat{K}|F\rangle = \int_a^b dx\, \langle G|x\rangle\langle x|\hat{K}|F\rangle$$

[a, b are the limits of the eigenvalues $\hat{X}$ can have — i.e. the limits of the vector space.

Insert 1 again: $\int_a^b\int_a^b dx'dx\, \langle G|x\rangle\langle x|\hat{K}|x'\rangle\langle x'|F\rangle$

$$\langle G|\hat{K}|F\rangle = \int_a^b\int_a^b dx'dx\, g^*(x)\, K_{xx'}\, f(x') = \int_a^b\int_a^b dx'dx\, g^*(x)\,\delta(x-x')\, i\frac{df}{dx'} \quad (1)$$

similarly $\langle F|\hat{K}|G\rangle^* = \left(\int_a^b\int_a^b dx'dx\, f^*(x)\, K_{xx'}\, g(x')\right)^* = \left(\int_a^b\int_a^b dx'dx\, f^*(x)\,\delta(x-x')\, i\frac{dg}{dx'}\right)^* \quad (2)$

Does (1) = (2) ?

(1)

$$\int_a^b dx\, g^*(x) \frac{-i\, df}{dx}$$

$$= -i\, g^*(x) f(x) \Big|_a^b + i \int_a^b dx \frac{dg^*}{dx} f(x)$$

(2)

$$\left(\int_a^b dx\, f^*(x) \frac{-i\, dg}{dx} \right)^*$$

$$= i \int_a^b dx\, f(x) \frac{dg^*}{dx}$$

Clearly equal if $g^*(b)f(b) = g^*(a)f(a)$

So we solve this by restricting the vector space to those vectors that:

Hilbert Space
- Go to zero at endpoints
- Are defined over periodic variables
- Are defined to $\pm\infty$ and oscillate — see Shankar page 66

Eigenvalues of $\hat{K} = -i\hat{D}$:

Take $|k\rangle$ as eigenvector of $\hat{K}$: $\hat{K}|k\rangle = k|k\rangle$:

$$\langle x|\hat{K}|k\rangle = k\langle x|k\rangle$$

$$\int dx' \langle x|\hat{K}|x'\rangle\langle x'|k\rangle = k\langle x|k\rangle = k\psi_k(x)$$

$$= -i \int dx'\, \delta'(x'-x)\langle x'|k\rangle = k\psi_k(x)$$

$$= -i \frac{d\psi_k}{dx} = k\psi_k(x)$$

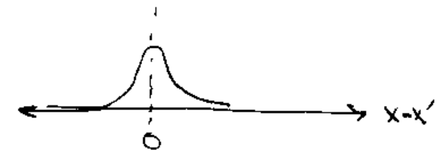
$$\rightarrow \psi_k(x) = \langle x|k\rangle = Ae^{ikx}$$

Note that $\langle k|k'\rangle = \delta(k,k')$ so $\langle k|k\rangle = \infty$.

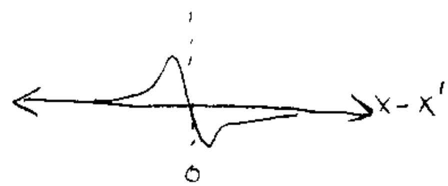
• Why is k real? If complex, then $\langle x|k\rangle$ blows up at $+$ or $-\infty$, and thus $|k\rangle$ not a vector in the Hilbert space. So restricting $|k\rangle$ to Hilbert space does <u>indeed</u> keep $\hat{K}$ Hermitian.

A diversion: What is the discrete analogue of $\hat{D}$?

Delta fn is limit of continuous, even function:



$\hat{D}$ is limit of its derivative:



→ zero on diagonal
→ positive "just" off-diagonal to left
→ negative "just" off-diagonal to right
→ zero elsewhere

Basis transformation: Take a vector $|G\rangle$, whose rep. in $\hat{X}$ basis is $\langle x|G\rangle = g(x)$. What is rep. in $\hat{K}$ basis $\langle k|G\rangle$?

Use completeness identity $1 = \int dx\, |x\rangle\langle x|$:

$$\langle k | G \rangle = \int dx \, \langle k | x \rangle \langle x | G \rangle$$
$$= \int dx \, A e^{ikx} \, g(x)$$

Similarly

$$\langle x | G \rangle = \int dk \, \langle x | k \rangle \langle k | G \rangle$$
$$= \int dk \, B e^{-ikx} \langle k | G \rangle$$

So $\langle x | G \rangle$ and $\langle k | G \rangle$ are Fourier transforms of one another.