LECTURE 5

A final comment on the 2-basis example from Monday:

"A" basis $|i_a\rangle$, $i=1,\dots,N$

"B" basis $|i_b\rangle$, $i=1,\dots,N$

Components of basis transformation operator $\hat{U}$:

in "A": $$\hat{U}_{ij} = \langle i_a|\Big(\sum_k |k_b\rangle\underbrace{\langle k_a|\Big)|j_a\rangle}_{=\delta_{kj}} = \langle i_a|j_b\rangle$$

in "B": $$\hat{U}_{ij} = \underbrace{\langle i_b|\Big(\sum_k |k_b\rangle}_{=\delta_{ik}}\langle k_a|\Big)|j_a\rangle = \langle i_a|j_b\rangle$$

This is the i^{th} component in the "A" basis of the j^{th} basis vector in the "B" basis.

$\boxed{\text{Read Example 1.8.6 (vector space analysis of classical motion)}}$

Infinite dimensions and continuous bases:

Showed on homework that a function of a continuous variable is a vector space.

Start with a vector space, with N dimensions. Number the dimensions $1, 2, \dots, N$. As we let $N \to \infty$, we get an infinite-dimension space. Could work with discrete basis vectors $|1\rangle, |2\rangle, \cdots, |\infty\rangle$. Another possibility is to keep the component index finite, but allow it to be non-integer.

so: basis $|i\rangle$, $i = 0,1,2,3,\dots,N$ 1+N-dim

$|i\rangle$, $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, N$ 1+2N-dim

$|x\rangle$, $x = (0,N)$ ∞-dim

$x = (-\infty, +\infty)$ also ∞-dim.

So our component index becomes a continuous variable x.

Properties of basis kets:

Call $|x\rangle$ the basis kets that are eigenvectors of operator $\hat{X}$, with eigenvalue X.

- $\hat{X}|x\rangle = x|x\rangle$
- $\hat{x}|x'\rangle = x'|x'\rangle$ (so we're using the ket label to indicate the eigenvalue)
- $\langle x'|x''\rangle = \delta(x'-x'')$ Orthogonality (generalization of $\langle i|j\rangle = \delta_{ij}$)
- $\int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| = \mathbb{1}$ Completeness (generalization of $\sum_i |i\rangle\langle i| = \mathbb{1}$)

Consider a normalized vector $|V\rangle$ in this space (not an $\hat{X}$ eigenstate): $\langle V|V\rangle = 1$. Using completeness relation,

$$\langle V| \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| V\rangle = 1$$

$$\int_{-\infty}^{\infty} dx' \langle V|x'\rangle\langle x'|V\rangle = 1$$

$$\int_{-\infty}^{\infty} dx' |\langle x'|V\rangle|^2 = 1$$

$|\langle x|v\rangle|^2$ is a normalized probability density
$= |\psi_v(x)|^2$

$\langle x|v\rangle$ is wave function in QM.

Think of $\psi(x)$ as the continuous limit of the list of components of a vector in infinite dimensions.

$\hat{X}$ is an operator, and it should be diagonal in the $\hat{x}$ basis. Check: What are matrix elements?

- Can't "draw" a matrix since rows, columns indices are continuous. But can define matrix element as analogue to $A_{ij} = \langle i|\hat{A}|j\rangle$:

$$\langle x''|\hat{x}|x'\rangle = x\langle x''|x'\rangle$$
$$= x'\delta(x''-x') \quad \text{which } \underline{\text{is}} \text{ diagonal.}$$

Derivative operator:

Consider ket $|f\rangle$, such that $\langle x|f\rangle = f(x)$. (i.e. $|f\rangle$ not a basis state). Want to find an operator $\hat{D}$ such that $\langle x|\hat{D}|f\rangle = \frac{df}{dx}$.

Call $\hat{D}|f\rangle \equiv \left|\frac{df}{dx}\right\rangle$. What are matrix elements of $\hat{D}$?

$$\langle x|\hat{D}|x'\rangle = \frac{d}{dx}\langle x|x'\rangle = \frac{d}{dx}\delta(x-x')$$

(huh? Delta fn. has a derivative?)