LECTURE 43

Continuing with Dirac equation:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \left(c\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + \hat{\beta}mc^2\right)|\psi\rangle$$

where $\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}$ $\quad \hat{\beta} = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}$

So relativity gives us a 1st-order equation, but requires that spinors have 4 components instead of 2: $\begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \longleftrightarrow \psi$ (no ket since this is x-basis already)

The Dirac eqn yields conservation laws:

$$\int d^3\vec{r}\, \psi^\dagger \psi = \text{constant}, \quad \text{where } \psi^\dagger\psi = \sum_{\sigma=1}^{4} \psi_\sigma^* \psi_\sigma$$

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \vec{\nabla}\cdot\vec{j} = 0 \quad \text{where } j_i \equiv c\,\psi^\dagger \hat{\alpha}_i \psi$$

First, ask: what are solutions corresponding to an electron at rest?
In this case, $\vec{P}|\psi\rangle = 0$, so Dirac eq. reduces to

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{\beta}mc^2\, \psi$$

In position-rep., there are four solutions:

$$\psi^1 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \psi^2 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \psi^3 = e^{+i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \psi^4 = e^{+i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

2 have energy mc^2.

2 have energy $-mc^2$.

Now, add an electromagnetic potential $(\vec{A}, \phi)$:

Classically $\mathcal{H} = \sqrt{(\vec{p} - e\vec{A}/c)^2 c^2 + m^2c^4} + e\phi$

So in Dirac eqn, replace $\hat{\vec{p}} \to \hat{\vec{p}} - e\vec{A}/c \equiv \vec{\pi}$

$$i\hbar \frac{\partial \psi}{\partial t} = \left(c\hat{\vec{\alpha}} \cdot \hat{\vec{\pi}} + \hat{\beta} mc^2 + e\phi \right) \psi$$

Can obtain electron spin, magnetic moment by examining non-relativistic limit. Take $\phi = 0$ for simplicity, and work to order $\left(\frac{v}{c}\right)^2$:

Look for eigenstates $\psi(t) = e^{-iEt/\hbar} \psi(0)$:

$$E\psi = \left(c\hat{\vec{\alpha}} \cdot \hat{\vec{\pi}} + \hat{\beta} mc^2 \right) \psi$$

Now write $\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ where χ, Φ are 2-component spinors.

Now put in forms for $\hat{\vec{\alpha}}, \hat{\beta}; \psi$: $\left(i\hbar \left(\frac{-iE}{\hbar} \right) - c\hat{\vec{\alpha}} \cdot \hat{\vec{\pi}} - \hat{\beta} mc^2 \right) \psi = 0$

$$\begin{pmatrix} E - mc^2 & -c\hat{\vec{\sigma}} \cdot \hat{\vec{\pi}} \\ -c\hat{\vec{\sigma}} \cdot \hat{\vec{\pi}} & E + mc^2 \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix} = 0$$

Separate eqns:
$$(E - mc^2)\chi = c\hat{\vec{\sigma}} \cdot \hat{\vec{\pi}} \Phi$$
$$(E + mc^2)\Phi = c\hat{\vec{\sigma}} \cdot \hat{\vec{\pi}} \chi$$

Solve 2^1 for Φ: $\Phi = \dfrac{c\hat{\vec{\sigma}} \cdot \hat{\vec{\pi}}}{E + mc^2} \chi$.

In nonrelativistic limit, $E \approx mc^2$, so

$$\Phi \approx \frac{\vec{\sigma}\cdot\vec{\pi}}{2mc}\chi$$

But numerator has order: σ order 1 (eigenvalues ± 1)
$\vec{\pi}$ order $P \ll mc$ in nonrel. limit

so for nonrel. particle $\Phi \ll \chi$: χ called large component
Φ small component.

Fortunate behavior:

- Negative energy stuff not relevant to low-energy dynamics
- 2-component spinor describes low-energy behavior

Now rewrite eqn of motion for χ alone: $E_s \equiv E - mc^2$

$$E_s\chi = c\,\hat{\vec{\sigma}}\cdot\hat{\vec{\pi}}\,\Phi = c\,\hat{\vec{\sigma}}\cdot\hat{\vec{\pi}}\,\frac{\hat{\vec{\sigma}}\cdot\hat{\vec{\pi}}}{2mc}\chi = \frac{(\hat{\vec{\sigma}}\cdot\hat{\vec{\pi}})(\hat{\vec{\sigma}}\cdot\hat{\vec{\pi}})}{2m}\chi$$

= Pauli Equation.

Recall identity: $(\hat{\vec{\sigma}}\cdot\vec{A})(\hat{\vec{\sigma}}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\,\hat{\vec{\sigma}}\cdot(\vec{A}\times\vec{B})$

so $$E_s\chi = \frac{\hat{\vec{\pi}}\cdot\hat{\vec{\pi}} + i\,\hat{\vec{\sigma}}\cdot(\hat{\vec{\pi}}\times\hat{\vec{\pi}})}{2m}\chi$$

$$\vec{\pi}\cdot\vec{\pi} = \left(\hat{\vec{P}} - e\vec{A}/c\right)^2$$

$$\vec{\pi}\times\vec{\pi} = \underbrace{\cancel{\hat{\vec{P}}\times\hat{\vec{P}}}}_{0} - \frac{e}{c}\left(\hat{\vec{P}}\times\hat{\vec{A}} + \hat{\vec{A}}\times\hat{\vec{P}}\right) + \cancel{\left(\frac{e}{c}\right)^2\vec{A}\times\vec{A}} =$$

$$= -\frac{e}{c}\left(-i\hbar\vec{\nabla}\times\hat{\vec{A}}\right) = \frac{ie\hbar}{c}\hat{\vec{B}}$$

$$E_s \chi = \left[\frac{(\vec{P} - e\vec{A}/c)^2}{2m} - \frac{e\hbar}{2mc} \hat{\vec{\sigma}} \cdot \hat{\vec{B}} \right] \chi$$

Recall def'n of magnetic moment: $\hat{H} = -\vec{\mu} \cdot \vec{B}$

so $\vec{\mu} = \frac{e\hbar}{2mc} \hat{\vec{\sigma}} = \frac{e}{mc} \hat{\vec{S}}$

Defined g earlier so $\mu = \frac{q g}{2mc}$ $\Rightarrow$ $\underline{g = 2}$.