4410 LECTURE 42

## Intro to relativistic QM (Not on exam)

Would like to take the Schrödinger equation and adapt it to the relativistic energy, replacing the non-relativistic Hamiltonian.

$$E_{rel}^2 = c^2 p^2 + m^2 c^4$$

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

The obvious strategy is to turn this into an operator equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \sqrt{\hat{p}^2 c^2 + m^2 c^4}\,|\psi\rangle$$

or in position basis:

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}\;\psi$$

This gives us a headache: how to interpret the square root?
One choice: expand in powers. Now $\hat{H}$ is dependent on all orders of momentum! Might neglect higher terms — but if $p$ is small why bother with relativity?

Consider squaring both sides: no more square root.

$$-\hbar^2 \frac{\partial^2}{\partial t^2}|\psi\rangle = \hat{H}^2|\psi\rangle$$

in position space:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left(-\hbar^2 \nabla^2 c^2 + m^2 c^4\right)\psi$$

or, as usually written:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc^2}{\hbar}\right)^2\right]\psi = 0$$

The Klein-Gordon equation. This is the equation that describes free scalar particles. One immediate consequence is that there are negative energy solutions: if $E$ is an eigenvalue, then so is $-E$! Interpretation: antiparticles.

This conveniently allows description of particle number freedom too: "Vacuum" has all negative states filled, all positive empty. Exciting one of the neg energy states to positive corresponds to creating a particle-antiparticle pair!

Moving on to fermions: Try to write $\sqrt{p^2c^2 + m^2c^4}$ as square root of (something linear in $p)^2$:

$$\hat{p}^2c^2 + m^2c^4 = \left[c\left(\hat{\alpha}_x\hat{p}_x + \hat{\alpha}_y\hat{p}_y + \hat{\alpha}_z\hat{p}_z\right) + \beta mc^2\right]^2$$

$$= \left(c\,\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + \hat{\beta}mc^2\right)^2$$

Match:

$$c^2\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + m^2c^4 = \left[c^2\left(\hat{\alpha}_x^2\hat{p}_x^2 + \hat{\alpha}_y^2\hat{p}_y^2 + \hat{\alpha}_z^2\hat{p}_z^2\right) + \beta^2m^2c^4\right]$$

+ all the cross-terms…

$$\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad i\hbar\frac{\partial|\psi\rangle}{\partial t} = \left(c\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + \beta mc^2\right)|\psi\rangle$$