LECTURE 40

We have finished all the "essential" material, a little early!

Now move on to extra topics: Quantization of the electromagnetic field (maybe Dirac eqn later).

So far we've considered the response of a particle to an externally applied field from either a charge or an EM wave, and treated fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ as continuous, well-defined functions of $\vec{r}, t$. Only truly quantum result of radiation was Fermi's golden rule: EM energy can only be emitted/absorbed in quanta of $\hbar\omega$.

An experimental contradiction in this picture: Transition rates btw. eigenstates of $\hat{H}$ for an atom in an EM field:

$$R \propto \left|\langle f|\hat{H}'|i\rangle\right|^2 \quad \text{where} \quad \hat{H}' \propto \vec{A} \quad \text{vector potential.}$$

So in the absence of any external field, the transition rate should vanish and $|i\rangle$ should live forever. (if $\hat{H}'=0$ then $|i\rangle$ is an $\hat{H}$ eigenstate $\rightarrow \Delta E = 0 \rightarrow \Delta t = \infty \rightarrow$ stationary state). But non-ground states of atoms tend to decay on the nanosecond scale!

So what's wrong? Setting $\vec{E} = 0 = \vec{B} = \vec{A}$ forever violates the uncertainty principle if $\vec{A}$ is a quantum observable! We'll show that for free space, $\langle\vec{E}\rangle = 0$ but $\langle E^2\rangle \neq 0$.

Take vector potential $\vec{A}$ in Coulomb gauge, no charge:

$$\vec{\nabla}\cdot\vec{A} = 0 \qquad \left(\text{Still: } \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right.$$
$$\Phi = 0. \qquad \left.\vec{B} = \vec{\nabla}\times\vec{A}\right)$$

We'll find it easier to quantize $\vec{A}$ than $\vec{E}, \vec{B}$ since it has $\frac{1}{2}$ the components.
First, consider spatial coordinate $\vec{r}$: here this will not be a dynamical variable, but a parameter more like an index.

To quantize field, begin by writing $\vec{A}(\vec{r},t)$ as a Fourier transform:

$$\vec{A}(r,t) = \int \frac{d^3k}{(2\pi)^3}\left[\tilde{\vec{A}}(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{\vec{A}}^*(\vec{k})e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

where $\omega = ck$ — This term keeps $\vec{A}$ real.

and the $e^{i\omega t}$ factor enforces Maxwell's equations absent charge.

The primary benefit of working in Fourier k-space is that the gauge criterion is simple:

$$\vec{\nabla}\cdot\vec{A} = 0 \xrightarrow{\text{Fourier}} \vec{k}\cdot\tilde{\vec{A}}(\vec{k}) = 0.$$

So, for each $\vec{k}$, the FT'd vector potential has zero component along $\vec{k}$.
Define two axes that rotate with $\vec{k}$, and remain perp. to k: $\vec{e}_{1\vec{k}}, \vec{e}_{2\vec{k}}$
Since $\vec{A}\cdot$ "$\vec{e}_{3\vec{k}}$" $=0$, $\tilde{\vec{A}}(\vec{k}) = A_{1\vec{k}}\vec{e}_{1k} + A_{2\vec{k}}\vec{e}_{2\vec{k}}$. Now rewrite the $\vec{r}$-space vector potential as

$$\vec{A}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=1}^{2}\left[A_{\lambda\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + A^*_{\lambda\vec{k}}e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]\vec{e}_{\lambda\vec{k}}$$

where the λ's are polarization directions 1 and 2 perp. to k.
Now we want to upgrade these $A_{\lambda\vec{k}}$, $A^*_{\lambda\vec{k}}$ to quantum operators.
Take a Hamiltonian approach:

$$\mathcal{H}_{\text{classical}} = \int d^3r\, \frac{E^2+B^2}{8\pi} = \frac{1}{2\pi c^2}\int d^3k \sum_{\lambda=1}^{2}\omega^2 A^*_{\lambda\vec{k}}A_{\lambda\vec{k}}$$

substitute defn of E, B in terms of $\vec{A}$.

Now, introduce new **variables**:

$$Q_{\lambda\vec{k}} \equiv \frac{1}{\sqrt{4\pi c^2}}\left(A^*_{\lambda\vec{k}} + A_{\lambda\vec{k}}\right), \quad P_{\lambda\vec{k}} \equiv \frac{i\omega}{\sqrt{4\pi c^2}}\left(A^*_{\lambda k} - A_{\lambda\vec{k}}\right)$$

It turns out that Q, P are canonically conjugate variables for a given $\vec{k}, \lambda$. Can show this in two ways. Shankar: Lagrangian approach (with B^2 as "kinetic energy" term) with $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. Or, show that $P_{\lambda\vec{k}}$ and $Q_{\lambda\vec{k}}$ satisfy Hamilton's equations:

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = \frac{-\partial \mathcal{H}}{\partial q}$$

Attaching $e^{i\omega t}$ time dependence to $Q_{\lambda\vec{k}}$ and $P_{\lambda\vec{k}}$, can show easily that they satisfy Hamilton's eqns. Now, rewrite $\mathcal{H}$ in terms of new variables:

$$\mathcal{H} = \int d^3k \sum_{\lambda=1}^{2}\left(\frac{1}{2}P^2_{\lambda\vec{k}} + \frac{\omega^2}{2}Q^2_{\lambda\vec{k}}\right)$$

... which is in the form of a sum of harmonic oscillators for each $\lambda, \vec{k}$. Makes some sense, since EM waves in vacuum are a superposition of vibrational modes.

Now take the quantum step: $P \to \hat{P}$, $Q \to \hat{Q}$. Since these are canonically conjugate, we already know their commutator:

$$\left[\hat{Q}_{\lambda_1\vec{k}_1}, \hat{P}_{\lambda_2\vec{k}_2}\right] = i\hbar\, \delta_{\lambda_1\lambda_2}\, \delta(\vec{k}_1 - \vec{k}_2).$$

The deltas are necessary to keep polarization states, $\vec{k}$ vectors independent.

Given a harmonic oscillator Hamiltonian, we can construct creation and annihilation operators:

$$\hat{a}_{\lambda\vec{k}} = \sqrt{\frac{\omega}{2\hbar}}\,\hat{Q}_{\lambda\vec{k}} + \frac{i}{\sqrt{2\hbar\omega}}\hat{P}_{\lambda\vec{k}} = \sqrt{\frac{\omega}{2\pi\hbar c^2}}\,\hat{A}_{\lambda k}$$

$$\hat{a}^{\dagger}_{\lambda\vec{k}} = \sqrt{\frac{\omega}{2\hbar}}\,\hat{Q}_{\lambda\vec{k}} - \frac{i}{\sqrt{2\hbar\omega}}\hat{P}_{\lambda\vec{k}} = \sqrt{\frac{\omega}{2\pi\hbar c^2}}\,\hat{A}^{\dagger}_{\lambda k}$$

so aside from units, the Fourier transform of $\vec{A}$ gives us the creation & annihilation operators.

Now combine into the number operator as before: $\hat{N}_{\lambda\vec{k}} = \hat{a}^{\dagger}_{\lambda\vec{k}}\,\hat{a}_{\lambda\vec{k}}$

such that $\hat{H} = \int d^3k \sum_{\lambda=1}^{2} \hbar\omega\left(\hat{N}_{\lambda\vec{k}} + \frac{1}{2}\right)$

(Note that this is actually energy/volume.)

We can use these number operators $\hat{N}_{\lambda\vec{k}}$ to interpret <u>photons</u> as excitations of the electromagnetic field.

So now we can interpret the number operators $\hat{N}_{\lambda\vec{k}}$ as our complete set of commuting operators, and write down what the Hilbert space is: direct product of an infinite # of harmonic oscillator spaces! Other complete sets of commuting operators: $\{\hat{Q}_{\lambda\vec{k}}\}$ or $\{\hat{P}_{\lambda\vec{k}}\}$.

Eigenstates of the $\hat{N}_{\lambda\vec{k}}$ basis: described by integers such as:

$$|n_{1\vec{k}_a} n_{2\vec{k}_a} n_{3\vec{k}_b} n_{2\vec{k}_b} n_{1\vec{k}_c} n_{2\vec{k}_c} \cdots \rangle$$

where there's one n for each $\lambda\vec{k}$. This represents a state with $n_{1\vec{k}_a}$ photons in polarization 1 and wave number $\vec{k}_a$. $n_{\lambda\vec{k}}$ is called an occupation number. The states of different occupation # are related by the creation & annihilation operators $\hat{a}^{\dagger}_{\lambda\vec{k}}$, $\hat{a}_{\lambda\vec{k}}$:

$$\hat{a}^{\dagger}_{\lambda\vec{k}'} |n_{1\vec{k}_a} \cdots n_{\lambda\vec{k}'} \cdots \rangle = |n_{1\vec{k}_a} \cdots n_{\lambda\vec{k}'}+1 \cdots \rangle$$

$$\hat{a}_{\lambda\vec{k}'} |n_{1\vec{k}_a} \cdots n_{\lambda\vec{k}'} \cdots \rangle = |n_{1\vec{k}_a} \cdots n_{\lambda\vec{k}'}-1 \cdots \rangle$$

Now consider the no-photons state $|0\rangle$. We can create a state with one photon of wave # $\vec{k}$, pol. λ by operating with creation operator:

$$\hat{a}^{\dagger}_{\lambda\vec{k}}|0\rangle = |0,0,0,\underset{\uparrow\ \lambda\vec{k}^{\text{th}}\ \text{occupation \#}}{1},0,0\cdots\rangle$$

Finally, consider electromagnetic energy in the "free-space vacuum" state $|0\rangle$. Energy density:

$$\hat{H} = \int d^3k \sum_{\lambda=1}^{2} \hbar\omega\left(\hat{N}_{\lambda\vec{k}} + \frac{1}{2}\right)$$

But $\hat{N}_{\lambda\vec{k}}|0\rangle = 0$, so

$$\frac{E}{\text{vol.}} = \int d^3k \sum_{\lambda=1}^{2} \hbar\omega \cdot \frac{1}{2} = \infty .$$

So even the "zero-point energy" of the vacuum diverges!

Quantum field theory is plagued with divergences like this, but they do not prevent us from making useful calculations. This divergent energy cannot be utilized or measured — only energy differences above it can: energy can be traded between the field and matter particles by creation and annihilation of photons, leaving the zero-point (ground state) energy ~~[illegible]~~ as the unchangeable minimum energy residing in the EM field.