P4410 LECTURE 4

Eigenstuff continued:

Friday got to the characteristic equation for the eigenvectors (eigenstates) of a matrix (operator):

$$\det(\hat{\Omega} - \omega \mathbb{1}) = 0.$$

For an operator defined on an N-dimensional vector space, this determinant becomes an N^{th} order polynomial. Since an N^{th} order polynomial has N roots, we can say that an operator defined on an N-dimensional vector space has N (not necessarily distinct) eigenvalues.

⟹ Eigenvalues of Hermitian operators are real: ⟸

$$\hat{\Omega}|A\rangle = \omega|A\rangle \qquad \text{adjoint:} \quad \langle A|\hat{\Omega}^\dagger = \omega^* \langle A|$$

$$\langle A|\hat{\Omega}|A\rangle = \omega\langle A|A\rangle \qquad \langle \hat{A}|\hat{\Omega}^\dagger|A\rangle = \omega^*\langle A|A\rangle$$

But $\hat{\Omega}^\dagger = \hat{\Omega}$ if Hermitian,
so these are equal:

$$\omega\langle A|A\rangle = \omega^*\langle A|A\rangle$$

$$\omega = \omega^* \qquad \text{QED.}$$

⟹ There exists an orthonormal basis made up of eigenvectors of $\hat{\Omega}$ if $\hat{\Omega}$ is Hermitian. ⟸

Start with what we proved Friday: Eigenvectors of Hermitian $\hat{\Omega}$ are orthogonal if they have different eigenvalues:

$$\hat{\Omega}|i\rangle = \omega_i|i\rangle, \quad \hat{\Omega}|j\rangle = \omega_j|j\rangle$$

$$\langle i|j\rangle = 0 \quad \text{if} \quad \omega_i \neq \omega_j$$

Non-degenerate case:

Since we've shown that for an N-dim vector space:

- An operator $\hat{\Omega}$ has N eigenvalues
- The eigenvectors are mutually orthogonal ($\hat{\Omega}$ Hermitian)

we have all the conditions for an orthogonal basis formed from the set of eigenvectors of $\hat{\Omega}$.

In this basis, show that $\hat{\Omega}$ is represented by a diagonal matrix in this basis: (pick normalized $|i\rangle$)

$$\Omega_{ij} = \langle i|\hat{\Omega}|j\rangle = \omega_j \langle i|j\rangle = \omega_j \delta_{ij}$$

Degenerate case:

An example of a degenerate matrix: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = (1-\omega)(2-\omega)(1-\omega) - (2-\omega)$$

$$= (1-2\omega+\omega^2)(2-\omega) \quad +\omega-2$$

$$= \cancel{2} - 4\omega + 2\omega^2 \cancel{-\omega} + 2\omega^2 - \omega^3 + \cancel{\omega} \cancel{-2}$$

$$= \omega(-4 + 4\omega - \omega^2)$$

$$= -\omega(\omega-2)^2 = 0 \quad \text{if } \underline{\omega = 0 \text{ or } 2}$$

double root
$\rightarrow$ 2-fold degeneracy

Eigenvectors: First find $\omega = 0$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + c_1 \\ b_1 \\ a_1 + c_1 \end{pmatrix} \Rightarrow -a_1 = c_1,\ b_1 = 0 : \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvector with $\omega=2$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2a_2 \\ 2b_2 \\ 2c_2 \end{pmatrix} = \begin{pmatrix} a_2+c_2 \\ 2b_2 \\ a_2+c_2 \end{pmatrix}$$

so $a_2=c_2$, b_2 can take any value. This space of vectors is a <u>degenerate subspace</u>. Need to <u>pick</u> two orthogonal members of this space, along with the $\omega=0$ eigenvector above, to form a basis:

Could pick: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad |1\rangle, |2\rangle, |3\rangle$

or just as good: $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, |?\rangle, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad |1'\rangle, |2'\rangle, |3'\rangle$

Find $|2'\rangle$ by requiring: $\begin{pmatrix} a_{2'} \\ b_{2'} \\ c_{2'} \end{pmatrix}$ where $a_{2'}=c_{2'}$, $\langle 1'|2'\rangle$

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_{2'} \\ b_{2'} \\ a_{2'} \end{pmatrix} = 0 = 2a_{2'} + b_{2'} \quad \text{so} \quad |2'\rangle \Leftrightarrow c\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Normalize: $c^2(1+4+1) = 1$

so $|2'\rangle \Longleftrightarrow \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

"Diagonalizing a Hermitian operator" $\Longleftrightarrow$ "finding a basis where the operator is diagonal" $\Longleftrightarrow$ "finding the basis composed of its eigenvectors"

Basis transformations accomplished by <u>unitary</u> <u>operators</u>.

Eigenproperties of <u>unitary</u> operators:

Eigenvalues: $\hat{U}|v\rangle = u|v\rangle, \quad \langle v|\hat{U}^{+} = \langle v|u^{*}$

$$\Rightarrow \langle v|\hat{U}^{\dagger}\hat{U}|v\rangle = u^{*}u\langle v|v\rangle$$

but $\hat{U}^{\dagger}\hat{U} = 1$ so $\quad = \langle v|v\rangle$

$$\Rightarrow u^{*}u = 1$$

u is a <u>complex number with unit modulus.</u>

Eigenvectors of $\hat{U}$ are mutually orthogonal if no degeneracy (prove for yourself)

Guts of a basis transformation:

Take the "A basis": $|i_a\rangle,\ i_a = 1, \cdots, N$
"B basis": $|i_b\rangle,\ i_b = 1, \cdots, N$
} ordering of basis kets is arbitrary.

Transform $\hat{U} = \sum_{j=1}^{N} |j_b\rangle\langle j_a|, \quad \hat{U}^{\dagger} = \sum_{j=1}^{N} |j_a\rangle\langle j_b|$

$$\hat{U}|i_a\rangle = \sum_{j=1}^{N} |j_b\rangle\langle j_a|i_a\rangle = \sum_{j=1}^{N} |j_b\rangle\delta_{ij} = |i_b\rangle$$

Is $\hat{U}$ unitary?

$$\hat{U}^{\dagger}\hat{U} = \sum_{i,j=1}^{N}\left(|i_a\rangle\underbrace{\langle i_b|)(|j_b\rangle}_{=\delta_{ij}}\langle j_a|\right)$$

$$= \sum_{i,j} |i_a\rangle\delta_{ij}\langle j_a| = \sum_{i} |i_a\rangle\langle i_a| = 1 \quad \checkmark$$