The Born Approximation — potential as a perturbation.

Partial wave expansion is useful for smallish $kr_0 \lesssim 1$. Born approx. is applicable in the large kr_0 limit (and also in low energy regime under certain conditions).

Potential is treated as perturbation ⟶ use Fermi's golden rule, with initial state $|\vec{p}_i\rangle$ in momentum basis, final state $|\vec{p}_f\rangle$.

$$\text{Rate of transition } |\vec{p}_i\rangle \to \overset{\text{ANY}}{|\vec{p}_f\rangle} = \frac{2\pi}{\hbar}\int |\langle \vec{p}_f|\hat{V}|\vec{p}_i\rangle|^2 \,\delta(E_f - E_i)\, d^3p_f$$

pulls out $\frac{2m}{2p_i}$

$$= \frac{2\pi}{\hbar}\int dp_f\, d\Omega_f\, |\langle \vec{p}_f|\hat{V}|\vec{p}_i\rangle|^2\, \delta\left(\frac{p_f^2}{2m} - \frac{p_i^2}{2m}\right) p_f^2$$

$$= \frac{2\pi m}{\hbar}\int d\Omega_f\, p_i\, |\langle \vec{p}_f|\hat{V}|\vec{p}_i\rangle|^2$$

where $|\vec{p}_f| = |\vec{p}_i|$

Now $\sigma = \frac{\text{rate} \to \text{all } \vec{p}_f}{|j_{inc}|}$ where in Shankar's normalization:

$$\langle \vec{r}|\vec{p}_i\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\, e^{i\vec{p}_i\cdot\vec{r}/\hbar}$$

so $|j_{inc}| = \frac{\hbar k}{m(2\pi\hbar)^3}$ where $p_f = p_i = \hbar k$.

So $\sigma = (2\pi)^4 \hbar^2 m^2 \int d\Omega_f\, |\langle \vec{p}_f|\hat{V}|\vec{p}_i\rangle|^2$; $\frac{d\sigma}{d\Omega} = (2\pi)^4 \hbar^2 m^2 |\langle \vec{p}_f|\hat{V}|\vec{p}_i\rangle|^2$

Now evaluate matrix element:

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 \hbar^2 m^2}{((2\pi\hbar)^3)^2}\left|\int d^3r\, e^{-i\frac{\vec{p}_f\cdot\vec{r}}{\hbar}}\, V(\vec{r})\, e^{i\frac{\vec{p}_i\cdot\vec{r}}{\hbar}}\right|^2$$

$$= \frac{m^2}{(2\pi)^2\hbar^4} \left| \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r}) \right|^2$$

Where $\hbar\vec{q} = \vec{p}_f - \vec{p}_i$ $\quad \vec{q}$ is momentum transfer

Lots of things depend on $q^2 = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$!

Recalling $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 \implies f(\theta,\phi) = e^{i\delta}\sqrt{\frac{d\sigma}{d\Omega}}$

Turns out in this approx. $e^{i\delta} = -1$:

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2} \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})$$

So $f(\vec{q})$ is the Fourier transform of the potential into momentum transfer space — (in the Born approximation)

Now, consider central potential $V(\vec{r}) = V(r)$, $\vec{p}_i = p_i\vec{e}_z$.

$f(\theta,\phi) \longrightarrow f(\theta)$.

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2} \int r'^2\, d\cos\theta'\, d\phi'\, dr'\, e^{-iqr'\cos\theta'}\, V(r')$$

$$= \frac{-m}{\hbar^2} \int d\cos\theta'\, dr' \left[r'^2 e^{-iqr'\cos\theta'} V(r') \right]$$

$$= \frac{-2m}{\hbar^2 q} \int dr'\, V(r')\, r' \sin qr'$$

So this depends only on q

Calculation of $f(\theta)$ for Yukawa potential:

$$V(r) = g\frac{e^{-\mu_0 r}}{r} \qquad \text{where } \mu_0 = \frac{m_\pi c}{\hbar}$$

This is a model of the force between two nucleons at close range. $M \longrightarrow$ reduced mass of 2 protons $= \mu$

$$f(\theta) = \frac{-2\mu g}{\hbar^2 q} \int_0^\infty dr' \, e^{-\mu_0 r'} \sin q r'$$

$$= \frac{-2\mu g}{2\hbar^2 q i} \int_0^\infty dr' \left(e^{iqr'} - e^{-iqr'}\right) e^{-\mu_0 r'}$$

$$= \frac{-2\mu g}{\hbar^2(\mu_0^2 + q^2)}$$

so $\frac{d\sigma}{d\Omega} = \text{Const.} \cdot \frac{1}{\left(1 + \frac{q^2}{\mu_0^2}\right)^2}$

remember $q^2 = 4k^2 \sin^2\frac{\theta}{2}$.

so $\frac{d\sigma}{d\Omega} = \text{Const.} \cdot \frac{1}{\left(1 + 4\frac{k^2}{\mu_0^2}\sin^2\frac{\theta}{2}\right)^2}$