Slip into the time-dependent picture briefly to introduce the S-matrix:

$$\hat{S} = \hat{U}\left(t_i = -\infty, t_f = +\infty\right)$$ so it's the "infinite Δt" propagator.

So if the initial state is a plane wave $|\vec{p}_i\rangle$ then the final state is $\hat{S}|\vec{p}_i\rangle$.

So the probability of an incoming particle ending up in state $|\vec{p}_f + d\vec{p}_f\rangle = \int d\vec{p}_f \langle \vec{p}_f + d\vec{p}_f | \hat{S} | \vec{p}_i \rangle$

Since energy is still conserved, require that $p_f^2 = p_i^2$.

For momentum eigenstates this is usually a clumsy view. In partial wave basis, it can be simple: Angular momentum (+energy) conservation requires that $\hat{S}$ be diagonal in the $\hat{H}, \hat{L}^2, \hat{L}_z$ basis.

Recall scattering amplitude $f_k(\theta) = \sum_l (2l+1) a_l(k) P_l(\cos\theta)$

where $$a_l(k) = \frac{e^{2i\delta_l} - 1}{2ik}$$

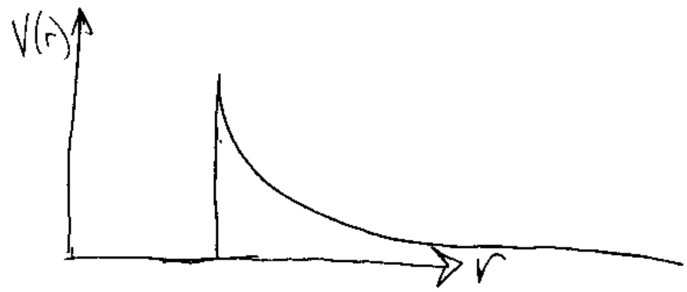
Let $S_l(k) \equiv \langle k, l_{out} | \hat{S} | k, l_{in} \rangle$ $\left(\begin{matrix} =0 \text{ if } k_{out} \neq k_{in} \\ \text{or } l_{out} \neq l_{in} \end{matrix}\right)$

Diagonal matrix element of $\hat{S}$.

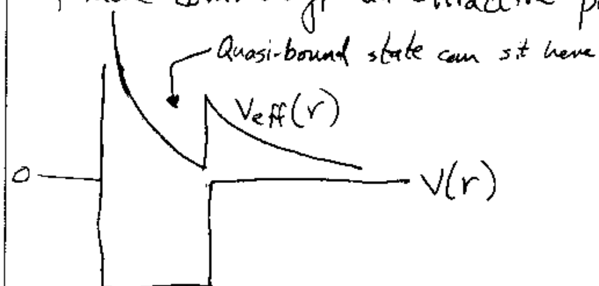
Note – absorbing $e^{il\pi}$ into $|k l_{out}\rangle$

So $S_l = \dfrac{\text{outgoing amplitude}}{\text{incoming amplitude}} = e^{2i\delta_l(k)}$ (Unitary as expected)

Now — consider a potential that allows quasi-bound states (aka metastable or tunneling states):



or, more commonly, an attractive potential with a centrifugal term:



Start by looking at a true bound state in $\lambda = 0$:

$$\psi(r) \sim S_0(k)\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r} \quad \text{for } k=0 \text{ scattering}$$

whereas at large distances, for a true bound state,

$$\psi(r) \sim \frac{e^{-\varkappa r}}{r}, \quad \varkappa \text{ real.}$$

So ratio of coefficients is ∞ for $k = i\varkappa$: a pole at a certain imaginary k! Note that no incoming wave appears; $\hat{S}$ not unitary. $E<0$ is not scattering.

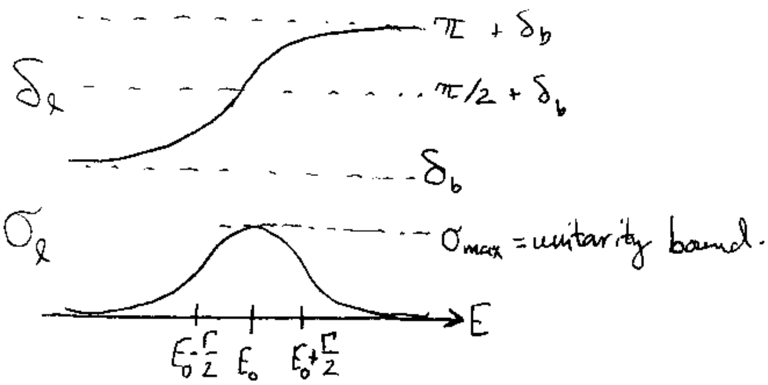
A quasi-bound state, however, has positive energy, so it can be probed by scattering experiments.

Scattering behavior of a quasibound state appears as a resonance.

$$\delta_\ell = \delta_b + \arctan\left(\frac{\Gamma/2}{E_0 - E}\right) \quad \text{where } \delta_b \cong n\pi$$

$$\sigma_\ell = \frac{4\pi}{k}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{(\Gamma/2)^2}{(E_0-E)^2+(\Gamma/2)^2}$$

This is called the Breit-Wigner formula:



where Γ is the width of the resonance.

This assumes $\frac{\Gamma}{E_0} \ll 1$. Otherwise need to take low-k dependence into account. $\lim_{k\to 0} \delta_\ell \propto (kr_0)^{2\ell+1}$ (See Shankar P. 550)

so can replace $\Gamma \to 2(kr_0)^{2\ell+1}\gamma$.

Now, confirm pole-like behavior:

$$S_\ell(k) = e^{2i\delta_\ell} = \frac{e^{i\delta_\ell}}{e^{-i\delta_\ell}} = \frac{1 + i\tan\delta_\ell}{1 - i\tan\delta_\ell} =$$

$$\frac{1 + i\,\frac{\Gamma/2}{E_0 - E}}{1 - i\,\frac{\Gamma/2}{E_0 - E}} = \frac{E - E_0 - i\Gamma/2}{E - E_0 + i\Gamma/2}$$

So there are poles at $E = E_0 - i\frac{\Gamma}{2}$. This is <u>not</u> a physical energy! But plug it into Schrödinger eqn anyway. Time dependence:

$$e^{-iEt/\hbar} \rightarrow e^{-i(E_0 - i\frac{\Gamma}{2})t/\hbar}$$

$$= e^{-iE_0 t/\hbar} e^{-\frac{\Gamma t}{2\hbar}}$$

Gives a "state" with exponential decay in time.

Resonance width $\Gamma \longleftrightarrow$ tunneling lifetime of quasibound state.