Continuing description of scattering:

So we've found in the external region, probability flux

$$\vec{J}_{scat} = \vec{e}_r \frac{1}{r^2}|f_k(\theta,\phi)|^2 \frac{\hbar k}{m}$$

so probability flux into $d\Omega$ is

$$R(d\Omega) = \left(\vec{J}_{scat} \cdot \vec{e}_r\right)\left(\frac{d\Omega}{r^2}\right)$$

$$= \frac{\hbar k}{m}|f_k(\theta,\phi)|^2 \, d\Omega$$

Differential cross-sec $\frac{d\sigma}{d\Omega}$ is $\frac{R(d\Omega)}{|J_{inc}|\,d\Omega} = \frac{R(d\Omega)}{\left|\frac{\hbar \vec{k}}{m}\right| d\Omega}$

so $$\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$$

Expansion in partial waves

Consider a plane wave state e^{ikz} scattering off a central potential.

- $V(r)$ is rotation invariant $\longrightarrow$ angular momentum is conserved.
- Incoming wave e^{ikz} has no ϕ dependence $\longrightarrow L_z = 0$
- So, outgoing ψ_{scat} must have $L_z = 0 \rightarrow$ no ϕ dependence. So we can take $f_k(\theta,\phi) = f_k(\theta)$

Our initial calculation of f_{scat} was in the context of $Y_{\ell m}$ components. Since we've shown that for a central potential, there is no ϕ-dependence, only $m=0$ components can be relevant:

$$f_k(\theta,\phi) = f_k(\theta) = \sum_\ell C_\ell Y_{\ell 0}(\theta)$$

Recall $Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$ P_ℓ = Legendre polynomials

$$\text{So} \quad f_k(\theta) = \sum_\ell \chi_\ell(k) P_\ell(\cos\theta)$$

Now, here's the key — we can expand e^{ikz} in $\ell, m=0$ eigenfunctions too! Incoming wave breaks down as:

$$e^{ikz} = e^{ikr\cos\theta} = \sum_\ell i^\ell (2\ell+1)\, j_\ell(kr) P_\ell(\cos\theta)$$

Each component has angular momentum $\ell(\ell+1)\hbar^2 = L^2$. Since the potential conserves angular momentum, the total outgoing flux must equal incoming flux for each ℓ component.

To facilitate future calcs, write components of $f_k(\theta)$ as

$$\chi_\ell(k) = (2\ell+1)\, a_\ell(k)$$

So we want to calculate the a_ℓ's. In practice, for small k (low energy) only the first few are significant — a quasiclassical argument gives $\ell_{max} \sim kr_0$ since $L = \vec{r}\times\vec{p} = \rho p_z = \hbar k \rho$.

Calculating a_ℓ: First understand how incoming wave decomposes.

At $kr \to \infty$, take $j_\ell(kr) \to \dfrac{\sin\left(kr - \frac{\ell\pi}{2}\right)}{kr}$

$$e^{ikz} \xrightarrow[kr\to\infty]{} \sum_\ell i^\ell (2\ell+1) \frac{\sin\left(kr - \frac{\ell\pi}{2}\right)}{kr} P_\ell(\cos\theta)$$

$$= \frac{1}{2ik} \sum_\ell i^\ell (2\ell+1) \left(\frac{e^{i\left(kr - \frac{\ell\pi}{2}\right)}}{r} - \frac{e^{-i\left(kr - \frac{\ell\pi}{2}\right)}}{r} \right) P_\ell(\cos\theta)$$

$$= \frac{1}{2ik} \sum_\ell (2\ell+1) \left[\frac{e^{ikr}}{r} - \frac{e^{-i(kr - \ell\pi)}}{r} \right] P_\ell(\cos\theta) \quad \left(\text{using } e^{-i\ell\frac{\pi}{2}} = i^{-\ell}\right)$$

(Typo in Shankar 19.5.8)

This is a superposition of incoming & outgoing waves of <u>same</u> modulus, phase shift of $\ell\pi$. In absence of scattering, this is what a plane wave looks like in spherical coordinates. Since the potential conserves angular momentum, $\left|\text{amplitude}\right|$ of $\frac{e^{ikr}}{r}$ and $\left|\frac{e^{-ikr}}{r}\right|$ waves <u>at each ℓ</u> can't change. Scattering can only be manifested as a phase shift at each ℓ:

$$\frac{e^{-i(kr - \ell\pi)}}{r} \longrightarrow \frac{e^{-i[kr - \ell\pi + \delta_\ell(k)]}}{r}$$

Total wave $\psi_k = \psi_{inc} + \psi_{scat}$ becomes:

$$\psi_k(\vec{r}) \underset{kr\to\infty}{=} \sum_\ell A_\ell \frac{e^{i\left(kr - \ell\frac{\pi}{2} + \delta_\ell\right)} - e^{-i\left(kr - \ell\frac{\pi}{2} + \delta_\ell\right)}}{r} P_\ell(\cos\theta)$$

Since incoming wave is unaffected by scattering, coeff's must match:

$$A_\ell = \frac{2\ell+1}{2ik}\, e^{i\left(\frac{\pi}{2}+\delta_\ell\right)}$$

So
$$\psi_k = \frac{1}{2ikr}\sum_\ell (2\ell+1)\left[e^{ikr}e^{2i\delta_\ell} - e^{-i(kr-\ell\pi)}\right]P_\ell(\cos\theta)$$

$$= e^{ikz} + \left[\sum_\ell (2\ell+1)\left(\frac{e^{2i\delta_\ell}-1}{2ik}\right)P_\ell(\cos\theta)\right]\frac{e^{ikr}}{r}$$

so scattering amplitude $a_\ell(k) = \dfrac{e^{2i\delta_\ell}-1}{2ik} = \dfrac{e^{i\delta_\ell}\sin\delta_\ell}{k}$

So
$$f_k(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) \qquad (*)$$

and
$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{1}{k^2}\left|\sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)\right|^2$$

Not trivial because of phases → cross-terms. But P_ℓ orthogonal, so:

$$\sigma = \int d\Omega\, |f|^2 = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

So σ is sum of partial cross-sections for each ℓ component:

$$\sigma = \sum_\ell \sigma_\ell \quad \text{where} \quad \sigma_\ell = \frac{4\pi}{k^2}\sin^2\delta_\ell$$

Unitarity bound at $\sin^2\delta_\ell = 1$.

Now, take $(*)$ and break into real, imaginary parts:

$$\mathrm{Im}(f_k(\theta)) = \frac{1}{k}\sum_\ell (2\ell+1)\sin^2\delta_\ell P_\ell(\cos\theta)$$

But recall $P_\ell(1) = 1 = P_\ell(\cos 0)$ so

$$\mathrm{Im}(f_k(0)) = \frac{1}{k}\sum_\ell (2\ell+1)\sin^2\delta_\ell = \frac{k}{4\pi}\sigma$$

so $\sigma = \frac{4\pi}{k}\mathrm{Im}(f_k(0))$ ← The optical theorem.

OK, now a calculation: hard sphere scattering:

$$V(r) = \infty \quad r < r_0, \quad = 0 \quad r > r_0.$$

Know that wave function must vanish as $r \to r_0$ from outside.
Only free particle solutions survive outside:

$$R(r) = \sum_\ell R_\ell(r) = \sum_\ell \left[A_\ell j_\ell(kr) + B_\ell n_\ell(kr)\right]$$

where coefficients match so $\sum_\ell R_\ell(r_0) = 0$.

Since there's only one solution, try setting individual $R_\ell(r_0) = 0$.
If it works, we're done.

$$\frac{B_\ell}{A_\ell} = \frac{-j_\ell(kr_0)}{n_\ell(kr_0)} \qquad \text{Looks OK.}$$

Radial function at large r:

$$R_\ell \to \frac{1}{kr}\left[A_\ell \sin\left(kr - \ell\frac{\pi}{2}\right) - B_\ell \cos\left(kr - \ell\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{kr}\sqrt{A_\ell^2 + B_\ell^2}\,\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$$

where $\delta_\ell = \arctan\left(\frac{-B_\ell}{A_\ell}\right) = \arctan\left(\frac{j_\ell(kr_0)}{n_\ell(kr_0)}\right)$

For $\ell = 0$: $\delta_0 = \arctan\left(\frac{\sin kr_0}{-\cos kr_0}\right) = -\arctan\left(\tan(kr_0)\right) = -kr_0$

In this case, we've treated the whole thing as a boundary value problem. One step further on homework: shell where the potential returns to 0 inside so some probability can penetrate.