Recall how we treated 1-D scattering: Assume incoming wave packet has narrow momentum distribution, take limit of momentum eigenstate:

(Shankar circular)

Target potential $V(x) = 0$ at $x \to \pm\infty$. In this limit:

Incoming wave Ae^{ikx} from $-\infty$

Outgoing Be^{-ikx} reflected, Ce^{ikx} transmitted

Only in $x \to -\infty$ Only in $x \to +\infty$

[Warning! + and − signs are reversed in Shankar 2nd ed., eq. 19.2.1]

Solve 1-D Schrödinger equation with these boundary conditions as a time-independent problem: find reflected fraction $R = \left|\frac{B}{A}\right|^2$, transmitted $\left|\frac{C}{A}\right|^2$.

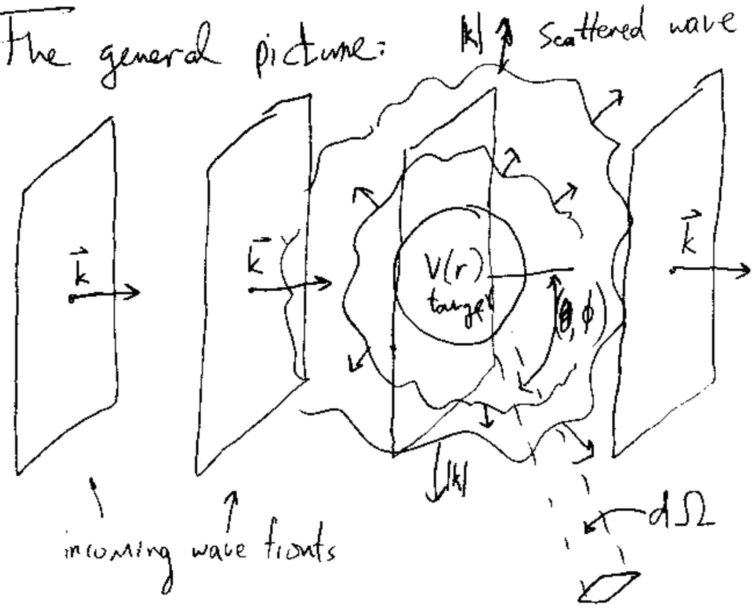
Differences in 3 dimensions:

- Need to consider incoming beam's <u>transverse</u> distribution. Typically take it to be uniform over range r_0 of target potential (i.e. where $V \to 0$).
- More to calculate than just R, T: $\dfrac{d\sigma(\theta,\phi)}{d\Omega} = \dfrac{\text{scatters into } d\Omega \text{ solid angle}}{\text{intensity of incoming beam.}}$
- No obvious distinction between "transmitted" forward and "reflected" backward regions — so we use for each momentum eigenstate, in limit $r \gg r_0$:

$$\psi_k = \psi_{inc} + \psi_{scat}$$

where $\psi_{inc} = e^{i\vec{k}\cdot\vec{r}}$ and ψ_{scat} is scattered wave.

The general picture:



About ψ_{sc}:

For $r \gg r_0$, $V = 0$ so form of ψ_{sc} is a solution to the free-particle Schrödinger eqn: Energy is conserved, so $|p| = |\hbar k|$:

$$\left(\nabla^2 + k^2\right)\psi_{sc} = 0$$

In spherical coordinates, free particle solutions are linear combinations of spherical Bessel and Neumann functions:

$$\psi_{sc} \xrightarrow[r \gg r_0]{} \sum_{lm} \left[A_l j_l(kr) + B_l n_l(kr)\right] Y_{lm}(\theta, \phi)$$

← Neumann fn. diverge at origin but we're only considering $r \gg r_0$.

Recall large-r behavior of j, n functions (Shankar 12.6.32):

$$j_l(\rho) \to \frac{1}{\rho}\sin\left(\rho - \frac{l\pi}{2}\right), \quad n_l(\rho) \to \frac{-1}{\rho}\cos\left(\rho - \frac{l\pi}{2}\right) \quad \text{as } \rho \to \infty.$$

Need to place constraint on ψ_{sc} that it be an outgoing wave only (e^{ikr}-type, no e^{-ikr} component). So normalization factors A_l, B_l have to set up a phase relation $(-n_l + i j_l)(C)$ at large r:

$$\frac{B_l}{A_l} = i \, .$$

$$\text{So } \psi_{sc} \underset{r\to\infty}{=} \sum_{lm}\left(-B_l \frac{e^{i\left[kr - \frac{l\pi}{2}\right]}}{kr}\right) Y_{lm}(\theta,\phi)$$

$$= \frac{+e^{ikr}}{kr} \sum_{lm} (-i)^l B_l Y_{lm}(\theta,\phi)$$

Which is a single radial fn. times a superposition of Y_{lm}'s.

Let $f_k(\theta,\phi) = \frac{1}{k}\sum_{lm} (-i)^l B_l Y_{lm}(\theta,\phi)$ "scattering amplitude"

And let incoming wave be in z direction e^{ikz} (divide out intensity).

Now $\psi = \psi_{inc} + \psi_{sc} \underset{r\to\infty}{=} e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}$

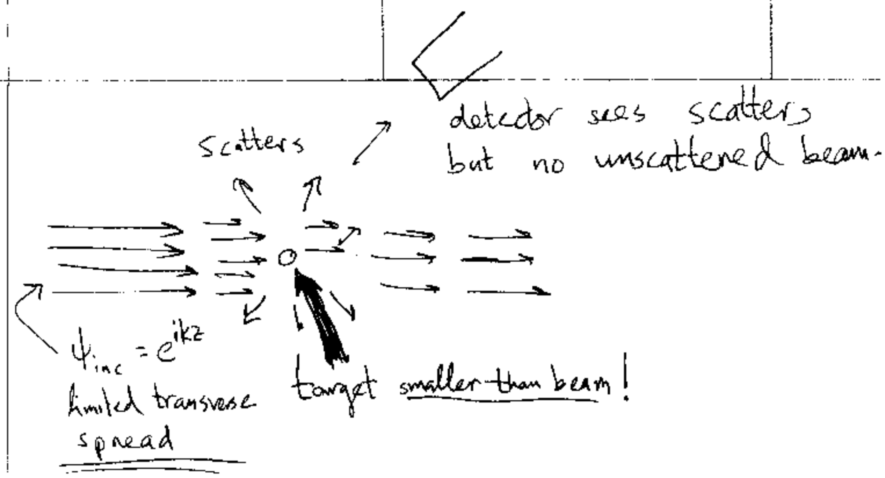
Now, look at the current density (aka probability current).

In 1-D: $|J| = \left|\frac{\hbar}{2mi}\left(\psi^* \frac{\partial\psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)\right|$

in 3-D: $|\vec{J}| = \left|\frac{\hbar}{2mi}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)\right|$

So if there were no scattering, $J = J_{inc} = \left|\frac{\hbar}{2mi}\left(ik\vec{e}_z + ik\vec{e}_z\right)\right| = \frac{\hbar k}{m}$

But if there is scattering, $\psi = \psi_{inc} + \psi_{scat}$ so J has cross-terms and we can't isolate J_{inc} and J_{sc}. So we make an <u>experimenter's</u> <u>assumption</u> that the beam is limited in transverse size and the detector is outside it.



so the detector is in a region where $\psi = \psi_{sc}$ only so

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi_{sc}^{*}\vec{\nabla}\psi_{sc} - \psi_{sc}\vec{\nabla}\psi_{sc}^{*}\right)$$

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \underbrace{\vec{e}_\theta \frac{\partial}{r\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}}_{\text{become small at large } r.}$$

Act on ψ_{sc}:
$$\frac{\partial}{\partial r} f_k(\theta,\phi)\frac{e^{ikr}}{r} = f_k(\theta,\phi)\, ik\frac{e^{ikr}}{r} + \mathcal{O}(r^2)$$

so
$$\vec{J}_{sc} = \frac{\vec{e}_r}{r^2}\left|f_k\right|^2 \frac{\hbar k}{m}$$