P4410 LECTURE 3

Component multiplication of operators:

$$(\hat{\Lambda}\hat{\Omega})_{ij} = \langle i|\hat{\Lambda}\hat{\Omega}|j\rangle = \langle i|\hat{\Lambda}\cdot\mathbb{1}\cdot\hat{\Omega}|j\rangle$$

(Use our identity $1 = \sum_i |i\rangle\langle i|$)

$$= \langle i|\hat{\Lambda}\left(\sum_k |k\rangle\langle k|\right)\hat{\Omega}|j\rangle$$

$$= \sum_k \langle i|\hat{\Lambda}|k\rangle\langle k|\hat{\Omega}|j\rangle$$

$$= \sum_k \Lambda_{ik}\Omega_{kj}$$ ← i.e. just matrix multiplication.

Special categories of operators:

- $\hat{A} = \hat{A}^\dagger$ Hermitian $(A_{ij} = A^*_{ji})$
- $\hat{A} = -\hat{A}^\dagger$ Anti-Hermitian

Note that any operator can be broken into Hermitian and anti-Hermitian parts:

$$\hat{A} = \frac{1}{2}(\hat{A} + \hat{A}^\dagger) + \frac{1}{2}(\hat{A} - \hat{A}^\dagger)$$

should remind you of:

$$f(x) = \frac{1}{2}\underbrace{\big(f(x) + f(-x)\big)}_{\text{even}} + \frac{1}{2}\underbrace{\big(f(x) - f(-x)\big)}_{\text{odd}}$$

- $\hat{U}^{-1} = \hat{U}^\dagger$ Unitary. (Generic unitary operators usu. written $\hat{U}$).

Note that $\hat{U}^\dagger\hat{U} = \hat{U}^{-1}\hat{U} = \mathbb{1}$.

Transformations between orthonormal bases are performed by unitary matrices. Since [redacted] inner products are basis-independent, unitary transformation must leave them unchanged. Check:

$$\text{Let } |V'\rangle = \hat{U}|V\rangle, \quad |W'\rangle = \hat{U}|W\rangle.$$

$$\langle W'| = (|W'\rangle)^\dagger = (\hat{U}|W\rangle)^\dagger = \langle W|\hat{U}^\dagger$$

$$\text{so } \langle W'|V'\rangle = \left(\langle W|\hat{U}^\dagger\right)\left(\hat{U}|V\rangle\right) = \langle W|\underbrace{\hat{U}^\dagger\hat{U}}_{=\mathbb{1}}|V\rangle = \langle W|V\rangle \checkmark$$

Another theorem: Rows or columns of a unitary matrix form the components of orthonormal vectors:

$$\hat{U}^\dagger\hat{U} = 1 \quad \text{: by components, } (\hat{U}^\dagger\hat{U})_{ij} = \delta_{ij}$$

$$= \sum_k U^\dagger_{ik} U_{kj} = \underbrace{\sum_k U^*_{ki} U_{kj}}$$

represents inner product of i^{th}, j^{th} column of $\mathbf{U}$.

Shankar makes a point of defining "active" and "passive" transformations.

$$\langle V|\hat{\Omega}|W\rangle \Rightarrow \underbrace{\langle V|(\hat{U}^\dagger\hat{\Omega}\hat{U})|W\rangle}_{\text{PASSIVE: transform the operator}} = \underbrace{(\langle V|\hat{U}^\dagger)\hat{\Omega}(\hat{U}|W\rangle)}_{\text{ACTIVE: transform the ket (vectors).}}$$

active, passive transforms are completely equivalent.

Eigenstuff:

in general, $\hat{\Omega}|V\rangle = |V'\rangle$. If $|V'\rangle$ is a scalar multiple of $|V\rangle$, then $\hat{\Omega}|V\rangle = \omega|V\rangle$ → eigenvector

↳ eigenvalue

• Two eigenvectors of Hermitian op. with different eigenvalues are orthogonal:

Let $\Omega|A\rangle = \alpha A$, $\Omega|B\rangle = \beta|B\rangle$. (easy to show for Hermitian!)

$$\left.\begin{aligned}\langle A|B\rangle &= \frac{1}{\beta}\langle A|\big(\Omega|B\rangle\big) = \\ &= \frac{1}{\alpha}\big(\langle A|\Omega\big)|B\rangle\end{aligned}\right\} \text{either } \frac{1}{\alpha} = \frac{1}{\beta} \text{ or } \langle A|B\rangle = 0.$$

Finding eigenvalues of a matrix:

$$\hat{\Omega}|V\rangle = \omega|V\rangle$$

$$\underbrace{(\hat{\Omega} - \omega\mathbb{1})}_{\text{singular}}|V\rangle = 0$$

⇒ Characteristic eq$^{\underline{n}}$.

$$\det(\hat{\Omega} - \omega\mathbb{1}) = 0$$