P3220 LECTURE 29

Showed Wed. the $\hat{L}_x, \hat{L}_y, \hat{L}_z$ operators in spherical coordinate basis:

$$\hat{L}_x \Longleftrightarrow i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y \Longleftrightarrow i\hbar\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z \Longleftrightarrow -i\hbar\frac{\partial}{\partial\phi}$$

Note that r-dependence has dropped out — $\vec{L}$ operators are associated with rotation only.

$$\hat{L}_\pm = +i\hbar\left[\left(\sin\phi \mp i\cos\phi\right)\frac{\partial}{\partial\theta} + \left(\cos\phi \pm i\sin\phi\right)\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

What are coordinate-basis eigenfunctions $\langle\vec{r}|l,m\rangle$?

- Must be independent of r
- Will obey $\hat{L}^2\phi_{l,m}(\theta,\phi) = \hbar^2 l(l+1)\phi_{lm}(\theta,\phi)$ 2

$$\hat{L}_z\phi_{l,m}(\theta,\phi) = \hbar^2 m\,\phi_{lm}(\theta,\phi) \quad 1$$

(Note – eigenfn's of $\hat{L}^2, \hat{L}_z$ not necessarily eigenfunctions of $\hat{H}$!)

Look for separable solutions — Alternative is to solve for $\hat{L}_+\phi_{ll} = 0$.

$$\phi_{l,m} = g(\theta)h(\phi)$$

1 $\hat{L}_z\, g(\theta)h(\phi) = -i\hbar\, g(\theta)\frac{\partial h}{\partial\phi} = m\hbar\, g(\theta)h(\phi) \Longrightarrow h(\phi) = e^{im\phi}$

(why not $Ae^{im\phi} + Be^{-im\phi}$?)

so for integer m: periodic in 2π

$\frac{1}{2}$ integer m: periodic in 4π (Whoa!)

(2) $\hat{L}^2 \phi_{lm} = \hat{L}^2\left(g(\theta)e^{im\phi}\right) = \hbar^2 l(l+1)\, \phi_{lm}$

$$\hbar^2 l(l+1)\, g(\theta)e^{im\phi} = \left(\hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z\right) g(\theta)e^{im\phi}$$

$$= \left[\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(\frac{-\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} + i\hbar^2\frac{\partial}{\partial\phi}\right]g(\theta)e^{im\phi}$$

Cancel the $\hbar^2$; expand out the $\hat{L}_z^2$, $\hat{L}_z$ terms:

$$l(l+1)\, g(\theta)e^{im\phi} = \left[e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(\frac{-\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\right]g(\theta)e^{im\phi} + (m^2 - m)\, g(\theta)e^{im\phi}$$

skipping steps:

$$= e^{im\phi}\left[-\frac{\partial}{\partial\theta}\left(\frac{\partial g}{\partial\theta} + mg\cot\theta\right) + (m-1)\cot\theta\left(\frac{\partial g}{\partial\theta} + mg\cot\theta\right) + m(m-1)\, g(\theta)\right]$$

Skipping more steps:

$$l(l+1)\, g(\theta) = \frac{-\partial^2 g}{\partial\theta^2} - \cot\theta\frac{\partial g}{\partial\theta} + m^2 g\csc^2\theta$$

mult. by $-\sin^2\theta$

$$-l(l+1)\sin^2\theta\, g(\theta) = \sin^2\theta\frac{\partial^2 g}{\partial\theta^2} + \sin\theta\cos\theta\frac{\partial g}{\partial\theta} - m^2 g$$

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \left[l(l+1)\sin^2\theta - m^2\right]g = 0$$

Solutions turn out to be:

$$g(\theta) = A\,P_{lm}(\cos\theta)$$

P_{lm} are the <u>associated Legendre polynomials</u>.

$$P_{lm}(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} \underbrace{P_l(x)}$$

↳ Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$g(\theta)h(\phi) = P_{lm}(\cos\theta)\,e^{im\phi}$ are the "spherical harmonics" $Y_{lm}(\theta,\phi)$.