P4410 LECTURE 21

Found Monday that $\hat{S}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2$ acts on uncoupled basis states $|m_1 m_2\rangle$ as: (spin $\frac{1}{2}$):

$$\hat{S}^2|++\rangle = 2\hbar^2|++\rangle \longrightarrow |++\rangle = |1\,1\rangle \text{ coup.}$$
$$\hat{S}^2|--\rangle = 2\hbar^2|--\rangle \longrightarrow |--\rangle = |1\,{-1}\rangle \text{ coup.}$$
$$\hat{S}^2|+-\rangle = \hbar^2(|+-\rangle + |-+\rangle)$$
$$\hat{S}^2|-+\rangle = \hbar^2(|+-\rangle + |-+\rangle)$$

Built remaining eigenstates of $\hat{S}^2, \hat{S}_z$ by:

$$|\text{asymm}\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$
$$|\text{symm}\rangle = \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle)$$

So $\hat{S}^2|\text{symm}\rangle = \frac{1}{\sqrt{2}}\hbar^2(|+-\rangle + |-+\rangle + |+-\rangle + |-+\rangle) = 2\hbar^2|\text{symm}\rangle$ ← $s=1$, $m=0$

$\hat{S}^2|\text{asymm}\rangle = \frac{1}{\sqrt{2}}\hbar^2(|+-\rangle + |-+\rangle - |+-\rangle - |-+\rangle) = 0.$ ← $s=0$ $m=0$

$$\Longrightarrow |1\,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$
$$|0\,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$ ← This state is unaffected by rotations! Try building a rotation operator and see effect on this vs. $s=1$ states.

The $s=1$ states are called "triplets"
The $s=0$ antisymmetric state = "singlet"

This type of structure appears often in particle physics — other quantum #'s (like flavor) have spin-like algebra: π^+, π^0, π^- form an "isospin" triplet, u, d quarks a doublet, etc.

Notation here is that the direct product (uncoupled) basis is related to the <u>direct sum</u> of a singlet and triplet space:

$$\underset{\text{doublet}}{\frac{1}{2}} \otimes \underset{\text{doublet}}{\frac{1}{2}} = \underset{\text{singlet}}{0} \oplus \underset{\text{triplet}}{1}$$

$$2 \cdot 2 = 1 + 3$$

The "direct sum" space is a different beast — it's basically 2 vectors tacked onto each other to form a new vector with the sum of the 2 #'s of components:

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \\ e \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \quad \text{while} \quad \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \\ e \end{pmatrix} \longrightarrow \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \text{something with 6 components.}$$

Note that the two bases span the same Hilbert space:

$$\sum_{m_1 m_2} |m_1 m_2\rangle\langle m_1 m_2| = \sum_{s\, m} |s\, m\rangle\langle s\, m| = 1.$$

So we can expand ~~either~~ any ket in either basis.
The transformation uses the matrix of Clebsch-Gordan coefficients: $\langle m_1 m_2 | s\, m\rangle$ which we have already calculated:

$\langle ++\|1\ 1\rangle = 1$	$\langle +-\|1\,1\rangle = 0$	$\langle -+\|1\,1\rangle = 0$	$\langle --\|1\,1\rangle = 0$
$\langle ++\|1\ 0\rangle = 0$	$\langle +-\|1\,0\rangle = \frac{1}{\sqrt{2}}$	$\langle -+\|1\,0\rangle = \frac{1}{\sqrt{2}}$	$\langle --\|1\,0\rangle = 0$
$\langle ++\|1\ {-1}\rangle = 0$	$\langle +-\|1\,{-1}\rangle = 0$	$\langle -+\|1\,{-1}\rangle = 0$	$\langle --\|1\,{-1}\rangle = 1$
$\langle ++\|0\ 0\rangle = 0$	$\langle +-\|0\,0\rangle = \frac{1}{\sqrt{2}}$	$\langle -+\|0\,0\rangle = \frac{-1}{\sqrt{2}}$	$\langle --\|0\,0\rangle = 0$

Now, move on to the more general case — this isn't trivial but for the first few j_1, j_2 values it is tractable. But there's a reason people tend to look the answer up in a table!

Two-"particle" (or L, S of one particle) direct product basis:

$$|j_1\, j_2\, m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle \quad \text{satisfies}$$

$$\hat{\vec{J}}_i^2 |m_1 m_2\rangle = j_i(j_i+1)\hbar^2 |m_1 m_2\rangle$$ where we're dropping j_1, j_2 from the ket label since they never change.

Total angular momentum basis:

$$|j_1\, j_2\, j\, m\rangle \to |j\, m\rangle = \sum_{m_1, m_2} |m_1 m_2\rangle\langle m_1 m_2 | j\, m\rangle \quad \text{satisfies}$$

$$\hat{\vec{J}}^2 |j\, m\rangle = j(j+1)\hbar^2 |j\, m\rangle, \quad \hat{J}_z |j\, m\rangle = m\hbar\, |j\, m\rangle$$

Note that we have a choice of phases — and there are different conventions out there! First, we choose all coefficients to be real. Remaining choice involves where — signs appear.

We showed last semester that the following bounds apply: Triangle inequality

$$|j_1 + j_2| \geq j \geq |j_1 - j_2| \qquad \text{Pick } j_1 \geq j_2 \text{ to avoid absolute value requirement.}$$

$$\therefore \quad j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus (j_1 - j_2)$$

Problem is to express each allowed $|j\, m\rangle$ as a superposition of $|j_1 m_1\rangle \otimes |j_2 m_2\rangle = |m_1 m_2\rangle$ states.

Start from the top: $|j_{max}\ j_{max}\rangle$. Since only one $|m_1 m_2\rangle$ can have the right m_1+m_2, we can write this C-G coefficient trivially:

$$|j=j_{max}, m=j_{max}\rangle = |j_1\ j_2\rangle \longrightarrow C_{j_{max} m_{max}\, j_1 j_2} = 1. \text{ (phase free)}.$$

Choose this phase to be 1. All other phases are now determined

Now, consider the next state: $|j_{max}\ m_{max}-1\rangle$.

$$\hat{S}_- |j_{max}\ j_{max}\rangle = \hbar\sqrt{(j_{max}+j_{max})(j_{max}-j_{max}+1)}\, |j_{max}\ m_{max}-1\rangle$$

$$= \sqrt{2j_{max}}\,\hbar\, |j_{max}\ j_{max}-1\rangle$$

But we want this in the uncoupled basis, so use $|j_{max}\ j_{max}\rangle = |m_1=j_1, m_2=j_2\rangle$ and $\hat{S}_- = \hat{S}_{1-} + \hat{S}_{2-}$: (writing full direct product notation for clarity)

$$\sqrt{j_{max}}\sqrt{2}\,\hbar\, |j_{max}\ j_{max}-1\rangle = \hat{S}_{1-}|j_1\ j_1\rangle\otimes|j_2\ j_2\rangle + \hat{S}_{2-}|j_1\ j_1\rangle\otimes|j_2\ j_2\rangle$$

$$= \hbar\left[\sqrt{(j_1+j_1)(j_1-j_1+1)}\, |j_1\ j_1-1\rangle\otimes|j_2\ j_2\rangle + \sqrt{(j_2+j_2)(j_2-j_2+1)}\, |j_1\ j_1\rangle\otimes|j_2\ j_2-1\rangle\right]$$

$$= \sqrt{2}\,\hbar\left(\sqrt{j_1}\, |j_1\ j_1-1\rangle\otimes|j_2\ j_2\rangle + \sqrt{j_2}\, |j_1\ j_1\rangle\otimes|j_2\ j_2-1\rangle\right)$$

So we can use the above equation to calculate $\langle j_{max}\ j_{max}-1 \mid m_1 = j_1 - 1, m_2 = j_2 \rangle$ and $\langle j_{max}\ j_{max}-1 \mid m_1 = j_1, m_2 = j_2 - 1 \rangle$ — i.e. the second row of Clebsch-Gordan coefficients. We can keep lowering m until we reach $-j_{max} \longrightarrow$ this gives all $|j_{max}\ m\rangle$ states.

How to lower j? Consider state $|j_{max}-1, j_{max}-1\rangle$ i.e. top m state. $j_{max} = j_1 + j_2$ as before. Now, what uncoupled basis states can contribute? All with $m_1 + m_2 = j_1 + j_2 - 1$:

$|m_1\, m_2\rangle = |j_1\ j_2-1\rangle$ or $|j_1 - 1\ j_2\rangle$.

We can build $|j_{max}-1, j_{max}-1\rangle$ out of these two states by following constraints:

- Must be normalized, real coefficients (convention)
- Must be orthogonal to $|j_{max}\ j_{max}-1\rangle$ which we calculated above
- Must have + sign on $m_1 = j_1$ ket (convention).

so
$$|j_{max}-1, j_{max}-1\rangle = \sqrt{\frac{j_1}{j_{max}}}\,|m_1 = j_1, m_2 = j_2 - 1\rangle - \sqrt{\frac{j_2}{j_{max}}}\,|m_1 = j_1 - 1, m_2 = j_2\rangle$$

... and we can lower m using $\hat{S}_- = \hat{S}_{1-} + \hat{S}_{2-}$ to get the remaining m states.

Third column: Top state: $|j_{max}-2, j_{max}-2\rangle$: Is orthogonal to <u>both</u> $m = j_{max}-2$ states (one from each of first two columns) and is a superposition of: $|m_1 = j_1, m_2 = j_2 - 2\rangle$ $|m_1 = j_1 - 2, m_2 = j_2\rangle$ $|m_1 = j_1 - 1, m_2 = j_2 - 1\rangle$.