P4410 LECTURE 18

Right now look at spin alone. Consider a <u>spinor field</u>: a spinor has 2 components, not 3. So, choose basis where S_z diagonal: $|+\rangle \Leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle \Leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$.

S_i's are Hermitian matrices:

$$\hat{S}_z \longleftrightarrow \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\hat{S}_x \longleftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$\hat{S}_y \Longleftrightarrow = ?$ Well, $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$ so

$$\hat{S}_y = \frac{\hbar}{2i}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]$$

$$= \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Define $\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$, etc. where

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Bigg\} \text{ PAULI SPIN MATRICES (very important!)}$$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hbar^2}{4}\sum_i \hat{\sigma}_i^2$$

$$\hat{\sigma}_x^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \quad \hat{\sigma}_y^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \quad \hat{\sigma}_z^2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

so $\hat{\sigma}_i^2 = \mathbb{1}$ for all $\hat{\sigma}_i$!

$\hat{S}^2 = \frac{\hbar^2}{4}(\mathbb{1}+\mathbb{1}+\mathbb{1}) = \frac{3}{4}\hbar^2\,\mathbb{1}$ so <u>every</u> state is an eigenstate of $\hat{S}^2$ with value $\frac{3}{4}\hbar^2 = \hbar^2 s(s+1) \Rightarrow s = \frac{1}{2}$.

A few properties of the Pauli matrices:

- They are Hermitian <u>and</u> unitary. So their eigenvalues are ± 1.

- Rotation-style commutation: $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_k$, or $\hat{\vec{\sigma}} \times \hat{\vec{\sigma}} = 2i\hat{\vec{\sigma}}$.

Rotations of spinor fields:

$$\hat{R}_z(\varepsilon) = \mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{S}_z = \mathbb{1} - \frac{i\varepsilon}{2}\hat{\sigma}_z \implies \hat{R}(\vec{\phi}_0) = e^{-\frac{i\vec{\phi}_0\cdot\hat{\vec{\sigma}}}{2}}$$

where, expanding:

$$\hat{R}(\vec{\phi}_0) = \mathbb{1} + \left(\frac{-i\phi_0}{2}\right)\left(\vec{e}_{\phi_0}\cdot\hat{\vec{\sigma}}\right) + \frac{1}{2!}\left(\frac{-i\phi_0}{2}\right)^2\left(\vec{e}_{\phi_0}\cdot\hat{\vec{\sigma}}\right)^2$$
$$+ \frac{1}{3!}\left(\frac{-i\phi_0}{2}\right)^3\left(\vec{e}_{\phi_0}\cdot\hat{\vec{\sigma}}\right)^3 + \cdots$$

But $$\left(\vec{e}_{\phi_0}\cdot\hat{\vec{\sigma}}\right)^2 = \left(\hat{\vec{\sigma}}\cdot\vec{e}_{\phi_0}\right)\left(\hat{\vec{\sigma}}\cdot\vec{e}_{\phi_0}\right)$$

$$= \underbrace{\vec{e}_{\phi_0}\cdot\vec{e}_{\phi_0}}_{=\mathbb{1}} + i\hat{\vec{\sigma}}\cdot\left(\underbrace{\vec{e}_{\phi_0}\times\vec{e}_{\phi_0}}_{0}\right) = \mathbb{1}$$

$$\hat{R}(\vec{\phi}_0) = \mathbb{1} + \frac{1}{2!}\left(\frac{-i\phi_0}{2}\right)^2 + \frac{1}{4!}\left(\frac{-i\phi_0}{2}\right)^4 + \cdots \quad \text{even terms}$$
$$+ \hat{\vec{\sigma}}\cdot\vec{e}_{\phi_0}\left[\frac{-i\phi_0}{2} + \frac{1}{3!}\left(\frac{-i\phi_0}{2}\right)^3 + \cdots\right] \quad \text{odd terms}$$

$$= \mathbb{1}\cos\frac{\phi_0}{2} - i\hat{\vec{\sigma}}\cdot\vec{e}_{\phi_0}\sin\frac{\phi_0}{2}$$

Now treat $\vec{\Phi}_0$ as rotation of (θ', ϕ'):

$$\vec{e}_{\phi_0} = \vec{e}_z \cos\theta' + \left(\vec{e}_x \cos\phi' + \vec{e}_y \sin\phi'\right)\sin\theta'$$

$$\hat{\vec{\sigma}} \cdot \vec{e}_{\phi_0} = \cos\theta' \hat{\sigma}_z + \left(\hat{\sigma}_x \cos\phi' + \hat{\sigma}_y \sin\phi'\right)\sin\theta'$$

$$= \begin{pmatrix} \cos\theta' & \sin\theta' e^{-i\phi'} \\ \sin\theta' e^{i\phi'} & -\cos\theta' \end{pmatrix}$$

Rotation by α about $\underline{z}$ done: $\theta' \to 0$, $\phi' \to \alpha$, $\phi_0 \to \alpha$

$$R(\alpha\vec{e}_z) = \begin{pmatrix} \cos\alpha/2 & 0 \\ 0 & -\cos\alpha/2 \end{pmatrix} - i\sin\frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

Check on an arbitrary state $|\psi\rangle = a|+z\rangle + b|-z\rangle$

$$R(\alpha\vec{e}_z)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae^{-i\alpha/2} \\ be^{i\alpha/2} \end{pmatrix}$$

Note that a 2π rotation does <u>not</u> bring you back to the original state! A **phase factor of -1 is** acquired! This **has** real experimental consequences.

See attached excerpt from Sakurai, "Modern Quantum Mechanics" (1985):

Let us now look at the time evolution of the state ket itself. Assuming that the initial ($t=0$) ket is given by (3.2.13), we obtain after time t

$$|\alpha, t_0=0; t\rangle = e^{-i\omega t/2}|+\rangle\langle+|\alpha\rangle + e^{+i\omega t/2}|-\rangle\langle-|\alpha\rangle. \quad (3.2.20)$$

Expression (3.2.20) acquires a minus sign at $t = 2\pi/\omega$, and we must wait until $t = 4\pi/\omega$ to get back to the original state ket with the same sign. To sum up, the period for the state ket is *twice* as long as the period for spin precession

$$\tau_{\text{precession}} = \frac{2\pi}{\omega}, \quad (3.2.21a)$$

$$\tau_{\text{state ket}} = \frac{4\pi}{\omega}. \quad (3.2.21b)$$

Neutron Interferometry Experiment to Study 2π Rotations

We now describe an experiment performed to detect the minus sign in (3.2.15). Quite clearly, if every state ket in the universe is multiplied by a minus sign, there will be no observable effect. The only way to detect the predicted minus sign is to make a comparison between an unrotated state and a rotated state. As in gravity-induced quantum interference, discussed in Section 2.6, we rely on the art of neutron interferometry to verify this extraordinary prediction of quantum mechanics.

A nearly monoenergetic beam of thermal neutrons is split into two parts—path A and path B; see Figure 3.2. Path A always goes through a magnetic-field-free region; in contrast, path B enters a small region where a static magnetic field is present. As a result, the neutron state ket going via path B suffers a phase change $e^{\mp i\omega T/2}$, where T is the time spent in the $\mathbf{B} \neq 0$ region and ω is the spin-precession frequency

$$\omega = \frac{g_n eB}{m_p c}, \quad (g_n \simeq -1.91) \quad (3.2.22)$$

for the neutron with a magnetic moment of $g_n e\hbar/2m_p c$, as we can see if we

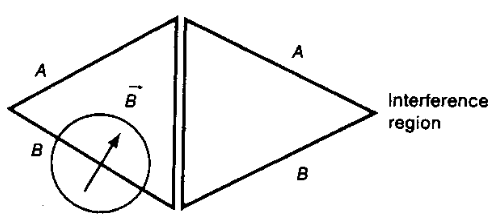

FIGURE 3.2. Experiment to study the predicted minus sign under a 2π rotation.

compare this with (3.2.17), which is appropriate for the electron with magnetic moment $e\hbar/2m_e c$. When path A and path B meet again in the interference region of Figure 3.2, the amplitude of the neutron arriving via path B is

$$c_2 = c_2(B=0)\, e^{\mp i\omega T/2}, \quad (3.2.23)$$

while the amplitude of the neutron arriving via path A is c_1, independent of $\mathbf{B}$. So the intensity observable in the interference region must exhibit a sinusoidal variation

$$\cos\left(\frac{\mp \omega T}{2} + \delta\right), \quad (3.2.24)$$

where δ is the phase difference between c_1 and c_2 ($B=0$). In practice, T, the time spent in the $B \neq 0$ region, is fixed but the precession frequency ω is varied by changing the strength of the magnetic field. The intensity in the interference region as a function of B is predicted to have a sinusoidal variation. If we call ΔB the difference in B needed to produce successive maxima, we can easily show that

$$\Delta B = \frac{4\pi\hbar c}{e g_n \lambda\!\!\!^{-} l}, \quad (3.2.25)$$

where l is the path length.

In deriving this formula we used the fact that a 4π rotation is needed for the state ket to return to the original ket with the same sign, as required by our formalism. If, on the other hand, our description of spin ½ systems were incorrect and the ket were to return to its original ket with the same sign under a 2π rotation, the predicted value for ΔB would be just one-half of (3.2.25).

Two different groups have conclusively demonstrated experimentally that prediction (3.2.25) is correct to an accuracy of a fraction of a percent.* This is another triumph of quantum mechanics. The nontrivial prediction (3.2.15) has been experimentally established in a direct manner.

Pauli Two-Component Formalism

Manipulations with the state kets of spin ½ systems can be conveniently carried out using the two-component spinor formalism introduced by W. Pauli in 1926. In Section 1.3 we learned how a ket (bra) can be represented by a column (row) matrix; all we have to do is arrange the expansion coefficients in terms of a certain specified set of base kets into a

*H. Rauch et al., *Phys. Lett.* **54A**, 425 (1975); S. A. Werner et al., *Phys. Rev. Lett.* **35** (1975), 1053.