P4410 LECTURE 17

- Problem set has a couple of spherical harmonics problems for review. See lecture notes from last semester (I'll post most relevant).

Exam review Sunday 4-6 pm, Duane G131

Today: spin ~~Tomorrow~~ Wednesday: Clebsch-Gordan coefficients.

Covered rotations for scalar field:

$$\langle \vec{r}|\psi\rangle = \psi(\vec{r})$$

$$\langle \vec{r}|\hat{R}_z(\varepsilon)|\psi\rangle = \langle x+\varepsilon y, y-\varepsilon x, z|\psi\rangle$$

Particles may have internal degrees of freedom that require them to be described by more than a single scalar field. Consider a vector field:

$$\langle \vec{r}|\psi\rangle = \psi_x(\vec{r})\vec{e}_x + \psi_y(\vec{r})\vec{e}_y + \psi_z(\vec{r})\vec{e}_z$$

Spatial dependence will rotate as above; consider only rotation of vector direction:

$$\hat{R}_z(\varepsilon)\vec{A} = (A_x - \varepsilon A_y)\vec{e}_x + (A_y + \varepsilon A_x)\vec{e}_y + A_z\vec{e}_z \ .$$

In matrix form:

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

or $\vec{\psi}' = \left[\mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{S}_z\right]\vec{\psi}$, where $\hat{S}_z \leftrightarrow i\hbar\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Similarly: $\hat{S}_x \leftrightarrow i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$\hat{S}_y \leftrightarrow i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

Properties of these matrices:

- Hermitian – obvious from transpose-conjugate
- Satisfy commutation relations $[\hat{S}_i, \hat{S}_j] = i\hbar \varepsilon_{ijk} \hat{S}_k$

Typically, use a basis where S_z is diagonal:

$$\hat{S}_z \leftrightarrow \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{S}_y \leftrightarrow \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\hat{S}_x \leftrightarrow \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

These are also Hermitian and satisfy same commutation relations. (Check it!) (Also, what is the basis transform operator $\hat{U}$ between the Cartesian and S_z bases?)

So, now treat a vector field (i.e. a "ket with components"). It is clear that infinitesimal rotations must simultaneously rotate the components of the vector and the coordinates over which the vector field is defined:

$$\hat{R}_z(\varepsilon)|\vec{\psi}\rangle = \left[1 - \frac{i\varepsilon}{\hbar}\left(\hat{L}_z + \hat{S}_z\right)\right]|\vec{\psi}\rangle \equiv \left(\mathbb{1} - \frac{i\varepsilon \hat{J}_z}{\hbar}\right)|\vec{\psi}\rangle$$

(Why $\hat{L}_z + \hat{S}_z$ not $\hat{L}_z\hat{S}_z$?)

Now consider the spatial dependence of a particle to be described by the ket $|\psi\rangle$ in a "scalar field" Hilbert space, and the spin direction by the ket $|\chi\rangle$ in a 3-dimension spin Hilbert space. Full dynamics of the particle is then described by a direct product space:

$$|\vec{\Psi}\rangle \longrightarrow |\psi\rangle \otimes |\chi\rangle$$

Where a basis could be formed from position eigenkets in $|\psi\rangle$ space and S_z eigenkets in $|\chi\rangle$ space:

$$\langle \vec{r}, \vec{\sigma} | \vec{\Psi} \rangle \Longrightarrow \langle \vec{r} | \psi \rangle \otimes \langle \vec{\sigma} | \chi \rangle$$
$$= \psi(\vec{r})\chi(\vec{\sigma})$$

Where $\hat{\vec{L}}$ generates rotations of ψ; $\hat{\vec{S}}$ generates rotations of χ.

Since rotation about an axis is now $\hat{R}(\vec{\Phi}_0) = \exp\left(\frac{-i\vec{\Phi}_0 \cdot \vec{J}}{\hbar}\right)$ where $\vec{J} = \vec{L} + \vec{S}$, follows that rotational invariance $\Longleftrightarrow [\hat{H}, \hat{\vec{J}}] = 0$.

Then, there are energy + ang. momentum eigenstates $|E\, j\, m\rangle$ such that

$$\hat{H}|E jm\rangle = E\,|E jm\rangle$$
$$\hat{J}^2|E j m\rangle = \hbar^2 j(j+1)|E jm\rangle$$
$$\hat{J}_z|E\, j\, m\rangle = \hbar m|E\, j\, m\rangle$$

in general: these are eigenstates of $\hat{L}^2$ and $\hat{S}^2$ but not of $\hat{L}_z$ and $\hat{S}_z$; transforming between jm and L_z, S_z eigenstates involves the Clebsch-Gordan coefficients.