P4410 LECTURE 16

Finishing up angular momentum eigenstuff:

Already found $\hat{L}_z|m\rangle = m\hbar|m\rangle$

Recall from last semester: $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, $[\hat{L}^2, \hat{L}_z] = 0$.

Define $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ so $\hat{L}_+\hat{L}_- = \hat{L}_x^2 + \hat{L}_y^2 + i[\hat{L}_y, \hat{L}_x] = \hat{L}_x^2 + \hat{L}_y^2 + \hbar\hat{L}_z$

Now $\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z$.

Consider $[\hat{L}_z, \hat{L}_\pm]|m\rangle = \hat{L}_z\hat{L}_\pm|m\rangle - \hat{L}_\pm\hat{L}_z|m\rangle = \pm\hbar\hat{L}_\pm|m\rangle$

$$\longrightarrow \hat{L}_z\left(\hat{L}_\pm|m\rangle\right) = \left(\pm\hbar\hat{L}_\pm + \hat{L}_\pm\hat{L}_z\right)|m\rangle$$

$$= \hat{L}_\pm\left(\pm\hbar + \hat{L}_z\right)|m\rangle$$

$$= \hbar(1\pm m)\left[\hat{L}_\pm|m\rangle\right]$$

So $\hat{L}_\pm|m\rangle$ is an $\hat{L}_z$ eigenstate with eigenvalue $(m\pm1)\hbar$.

Now, assume we are also in an eigenstate of $\hat{L}^2$; eigenvalue α:

$\hat{L}^2|\alpha,m\rangle = \alpha|\alpha,m\rangle$. Since $[\hat{L}^2, L_\pm] = 0$,

$$\hat{L}^2\left(\hat{L}_\pm|\alpha,m\rangle\right) = \hat{L}_\pm\hat{L}^2|\alpha,m\rangle$$

$$= \alpha\,\hat{L}_\pm|\alpha,m\rangle \quad \text{so } \hat{L}_\pm \text{ doesn't affect } \hat{L}^2.$$

Degeneracy of $\hat{L}^2$: How many m for a given "α"?

$$\langle\hat{L}^2\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle \geq \langle L_z^2\rangle.$$

$$\geq \hbar^2 m^2$$

So a maximum m exists for a given α:

$$\hat{L}_\pm \left| \alpha, \pm m_{max} \right\rangle = 0 . \qquad \hat{L}_\pm \text{ must annihilate state.}$$

Use this to find α:

$$\hat{L}^2 \left| \alpha, -m_{max} \right\rangle = \left(\hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z \right) \left| \alpha, -m_{max} \right\rangle = \alpha \left| \alpha, m_{-max} \right\rangle$$

$$= \underbrace{\hat{L}_+\hat{L}_- \left| \alpha, -m_{max} \right\rangle}_{\to 0} + \left(m^2\hbar^2 + m\hbar^2 \right) \left| \alpha, -m_{max} \right\rangle$$

$$= \hbar^2 \left(m_{max} + 1 \right) m_{max}$$

call $m_{max} = l$: $\quad \hat{L}^2 \left| l\, m \right\rangle = l(l+1)\hbar^2 .$

As you recall: Given m runs in increments of 1 from $-l$ to $+l$, l must be integer or $\frac{1}{2}$ integer.

Shankar gives a nice analysis in his "astute reader" discussion:
→ How can m take on $\frac{1}{2}$ integer values? Showed that $-i\hbar \frac{\partial}{\partial\phi}$ can only have eigenvalues $m\hbar$ for integer m.

Obviously, solution is that we didn't actually use $-i\hbar\frac{\partial}{\partial\phi}$ here, just the commutation relations among the $\hat{L}_x, \hat{L}_y, \hat{L}_z$ operators. So we've solved a more general problem.

$-i\hbar\frac{\partial}{\partial\phi}$ only describes the total angular momentum for a particle described by a scalar field ($\psi(r,\theta,\phi)$ in position basis).
A particle with spin can be described by a vector (or spinor or tensor) field: $\psi(\vec{r}) = \psi_x(\vec{r})\,\vec{e}_x + \psi_y(\vec{r})\,\vec{e}_y + \psi_z(\vec{r})\,\vec{e}_z$ (for spin 1)

When you rotate this system, more has to change than just one field: not only $\psi(\vec{r}) \to \psi(\vec{r}')$

but $\vec{\psi}(\vec{r}) \to \vec{\psi}'(\vec{r}) \to \vec{\psi}'(\vec{r}')$ — i.e. vector itself rotates as well as moving the position location where it is defined!

$\hat{L}_x, \hat{L}_y, \hat{L}_z$ are the generators for the coordinate rotations

$\hat{S}_x, \hat{S}_y, \hat{S}_z$ are the generators of the vector rotation — i.e. the "shuffling" of the components of the vector.

Total effect of a rotation: generator of infinitesimal rotation is $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$.

$\hat{\vec{S}}$ → spin angular momentum.

$\hat{\vec{L}}$ → orbital angular momentum