P4410 LECTURE 15

Finishing up $\hat{L_z}$:

In polar coordinates – matrix element

$$\langle\phi|\hat{L_z}|\phi'\rangle = -i\hbar\,\delta'(\phi-\phi')$$
$$= -i\hbar\,\delta(\phi-\phi')\frac{d}{d\phi'}$$

Check ϕ-space wave functions transform as expected under $\hat{R}_{\phi_0} = e^{\frac{-i\phi_0\hat{L_z}}{\hbar}}$:

$$\langle\phi|\hat{R}_{\phi_0}|\psi\rangle$$
$$= \int_0^{2\pi} d\phi' \langle\phi|e^{\frac{-i\phi_0\hat{L_z}}{\hbar}}|\phi'\rangle\langle\phi'|\psi\rangle$$
$$= \int_0^{2\pi} d\phi' \langle\phi|\phi'+\phi_0\rangle\langle\phi'|\psi\rangle$$
$$= \int_0^{2\pi} d\phi'\, \delta(\phi-\phi'-\phi_0)\langle\phi'|\psi\rangle$$
$$= \langle\phi-\phi_0|\psi\rangle = \psi(\phi-\phi_0) \longleftarrow \text{so this works.}$$

Eigenvalues/eigenstates of $\hat{L_z}$:

$$\hat{L_z}|l_z\rangle = l_z|l_z\rangle$$
$$\longrightarrow \langle\phi|\hat{L_z}|l_z\rangle = -i\hbar\frac{d}{d\phi}\langle\phi|l_z\rangle = l_z\langle\phi|l_z\rangle$$
$$\text{so}\quad \langle\phi|l_z\rangle \propto e^{\frac{il_z\phi}{\hbar}} \equiv \psi_{l_z}(\phi)$$

But since ϕ is periodic in 2π, $\psi_{l_z}(\phi+2\pi)$ must equal $\psi_{l_z}(\phi)$!
So $\frac{l_z}{\hbar}$ = integer required. Call integer m (magnetic quantum number)
Label the state accordingly.

so $\hat{L}_z|m\rangle = m\hbar|m\rangle$, $\langle\phi|m\rangle = \frac{1}{\sqrt{2\pi}}e^{im\phi}$

The $|m\rangle$ form a complete orthonormal basis, which spans the space of all continuous single-valued functions that satisfy $\psi(\phi+2\pi n)=\psi(\phi)$ for any ϕ, integer n. So we can make a completeness statement:

$$\sum_{m=\infty}^{\infty}|m\rangle\langle m| = 1.$$

Check Hermiticity of $\hat{L}_z$: $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

$$\hat{L}_z^{\dagger} = \hat{p}_y^{\dagger}\hat{x}^{\dagger} - \hat{p}_x^{\dagger}\hat{y}^{\dagger}$$

$$= \hat{p}_y\hat{x} - \hat{p}_x\hat{y} \quad \text{but } [\hat{p}_y,\hat{x}] = [\hat{p}_x,\hat{x}] \text{ so}$$

$$= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z$$

But — recall the nagging issue of Hermiticity of $\hat{p}$ depending on the endpoint terms vanishing after integration! This is also an issue for $\hat{L}_z$:

$$\langle\psi|\hat{L}_z|\psi\rangle = \int_0^{2\pi}d\phi\int_0^{2\pi}d\phi'\,\langle\psi|\phi\rangle\underbrace{\langle\phi|\hat{L}_z|\phi'\rangle}_{=-i\hbar\delta'(\phi-\phi')}\langle\phi'|\psi\rangle$$

$$= \int d\phi\,\psi^*(\phi)\left(-i\hbar\frac{d}{d\phi}\psi(\phi)\right)$$

Integrate by parts:

$$= \underbrace{-i\hbar\,|\psi(\phi)|^2\Big|_0^{2\pi}}_{} + i\hbar\int_0^{2\pi}d\phi\left[\frac{d}{d\phi}\psi(\phi)\right]^*\psi(\phi)$$

Pure imaginary term! Must vanish for Hermiticity:

$|\psi(2\pi)| = |\psi(0)|$ (i.e. ψ single-valued — this is a restriction on the Hilbert space.)

$\vec{L}$ in 3 dimensions:

– Treating $\hat{L}_i$ as generator of rotations in dimension i, the components $\hat{L}_i$ and $\hat{L}_j$ had better not commute! Rotating by x then y is not the same as rotating by y then x.

$$\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\, \varepsilon_{ijk} \hat{L}_k \quad \text{or} \quad \hat{\vec{L}} \times \hat{\vec{L}} = i\hbar \hat{\vec{L}}.$$

also. $[\hat{L}_i, \hat{L}^2] = 0$ with $\hat{L}^2 = \sum_i \hat{L}_i^2$.

Rotation by finite angle in 3d very similar to 1d:

$$\hat{R}(\vec{\phi}_0) = \exp\left(\frac{-i\,\vec{\phi}_0 \cdot \hat{\vec{L}}}{\hbar}\right)$$

Show $\vec{e}_\phi \cdot \hat{\vec{L}}$ generates rotations about axis parallel to $\vec{e}_\phi$ (unit vector): Start with $\hat{z}$ component:

$$R[\delta\phi\, \hat{e}_z]\, \hat{\vec{X}} = (\hat{x}' - \hat{y}'\delta\phi) + \vec{e}_y(\hat{y}' + \hat{x}'\delta\phi) + e_z \hat{z}$$

$$= \hat{\vec{X}} + \delta\phi\left(-\hat{y}'\vec{e}_x + \hat{x}'\vec{e}_y\right)$$

$$= \hat{\vec{X}} + \delta\phi\, \vec{e}_z \times \left(\vec{e}_x \hat{x}' + \vec{e}_y \hat{y}'\right)$$

$$= \hat{\vec{X}} + \delta\vec{\phi} \times \vec{X}'$$

Generalize: $R[\delta\vec{\phi}]\vec{X}' = \hat{\vec{X}} + \delta\vec{\phi} \times \vec{X} + \mathcal{O}(\delta\phi^2)$

then expect $\langle \vec{X}' | R(\delta\vec{\phi}) | \psi \rangle = \langle \vec{X}' - \delta\vec{\phi} \times \vec{X}' | \psi \rangle + \mathcal{O}(\delta\phi^2)$

$$\Rightarrow \langle \vec{x}' | \left(1 - \frac{i\,\delta\vec{\phi}\cdot\vec{L}}{\hbar}\right) | \psi \rangle = \langle x' | \psi \rangle - \delta\vec{\Phi} \times \vec{x}' \cdot \vec{\nabla}' \langle \vec{x}' | \psi \rangle + O(\delta\Phi^2)$$

$$\text{or } \delta\vec{\Phi} \cdot \langle \vec{x}' | \vec{L} | \psi \rangle \longleftrightarrow -i\hbar\,\delta\vec{\Phi} \times \vec{x}' \cdot \vec{\nabla}' \langle \vec{x}' | \psi \rangle$$

$$= i\,\delta\vec{\Phi} \cdot \left[\vec{x}' \times (-i\hbar \vec{\nabla}') \right] \langle \vec{x}' | \psi \rangle$$

$$= \delta\vec{\Phi} \cdot \langle \vec{x}' | \vec{x} \times \vec{p} | \psi \rangle$$