P 4410 LECTURE 14

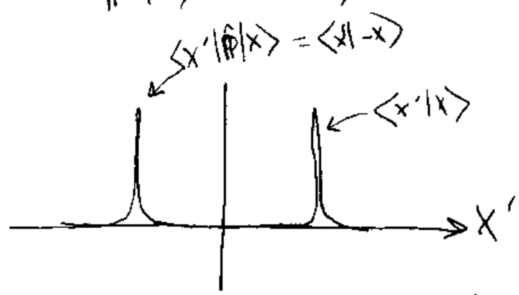
Discrete symmetries

→ Parity: Different notation for the parity operator:
$\hat{\mathbb{P}}$ (Liboff); $\hat{\Pi}$ (Shankar). I'll use $\hat{\mathbb{P}}$.

Classically: $q_i \to -q_i$
$p_i \to -p_i$

Since this is a discrete transformation, no infinitesimal generator.
In QM, only need to define its action on one basis:

$$\hat{\mathbb{P}}|x\rangle = |-x\rangle$$ ← Note: this is an $\hat{X}$ eigenstate w/ eigenvalue $-x$, not $-|x\rangle$.



Now, check the action of $\hat{\mathbb{P}}$ on momentum eigenstate $|p\rangle$:

$$\hat{\mathbb{P}}|p\rangle = \int dx\, \hat{\mathbb{P}}|x\rangle\langle x|p\rangle$$

$$= \int dx\, |-x\rangle e^{ipx/\hbar}$$

$$= \int dx\, |-x\rangle e^{i(-p)(-x)/\hbar}$$

$$= \int dx\, |-x\rangle\langle -x|-p\rangle$$

$$= |-p\rangle \quad \text{as expected classically}$$

So, of course, $\langle x|\hat{\mathbb{P}}|\psi\rangle = \int dx' \langle x|\hat{\mathbb{P}}|x'\rangle\langle x'|\psi\rangle$

$$= \int dx' \langle x | -x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \,\delta(x-(-x'))\,\psi(x')$$

$$= \psi(-x)$$

How to define parity invariance of $\hat{H}$:

$$\hat{\mathbb{P}}^\dagger \hat{H} \hat{\mathbb{P}} = \hat{H}$$

What is $\hat{\mathbb{P}}^\dagger$?

Easy to show $\hat{\mathbb{P}}$ is Hermitian (and unitary) since $\hat{\mathbb{P}}^2|x\rangle = |--x\rangle = |x\rangle \Rightarrow \hat{\mathbb{P}}$ has eigenvalues ± 1.

So if $\hat{\mathbb{P}}\hat{H}\hat{\mathbb{P}} = \hat{H}$

then $\hat{H}(\hat{x},\hat{p}) = \hat{H}(\hat{\mathbb{P}}\hat{x}\hat{\mathbb{P}}, \hat{\mathbb{P}}\hat{p}\hat{\mathbb{P}})$

$$= \hat{H}(-\hat{x},-\hat{p})$$

Note: if $\hat{H}$ is parity-invariant, then $[\hat{\mathbb{P}},\hat{H}] = 0$:

$\hat{\mathbb{P}}\hat{H} - \hat{H}\hat{\mathbb{P}} = \hat{\mathbb{P}}\underbrace{\hat{\mathbb{P}}\hat{H}\hat{\mathbb{P}}}_{} - \hat{H}\hat{\mathbb{P}} = 0$. So if $\hat{H}$ is parity invariant, can create a basis of simultaneous eigenstates of $\hat{H}$, $\hat{\mathbb{P}}$: parity is conserved.

Rotational symmetry:

Consider the operator $\hat{R}_\phi$ which rotates about the z axis: $\hat{R}_\phi|xy\rangle = |x'y'\rangle$ such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $$\hat{R}_\phi|xy\rangle = |x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi\rangle$$

Take $\phi \to \varepsilon$ infinitesimal. To 1st order in ε. $\cos\varepsilon = 1$, $\sin\varepsilon = \varepsilon$

$$\hat{R}_\varepsilon|xy\rangle = |x - \varepsilon y,\ y + \varepsilon x\rangle$$

Matrix element: $\langle x'y'|\hat{R}_\varepsilon|xy\rangle =$

$$= \delta(x' - (x - \varepsilon y))\,\delta(y' - (y + \varepsilon x))$$

Treat a state $|\psi\rangle$: $\langle xy|\hat{R}_\varepsilon|\psi\rangle = \psi(x + \varepsilon y,\ y - \varepsilon x)$

To first order: $= \psi(x,y) + y\varepsilon\frac{d\psi}{dx} - x\varepsilon\frac{d\psi}{dy} + \mathcal{O}(\varepsilon^2)$

Treat $\hat{R}_\varepsilon = \mathbb{1} - \frac{i\varepsilon\hat{G}}{\hbar} + \mathcal{O}(\varepsilon^2)$

so $$\langle xy|\frac{-i\varepsilon\hat{G}}{\hbar}|\psi\rangle = y\varepsilon\frac{d\psi}{dx} - x\varepsilon\frac{d\psi}{dy}$$

$\boxed{\text{But } \langle xy|\hat{L}_z|\psi\rangle = -i\hbar\left(x\frac{d\psi}{dy} - y\frac{d\psi}{dx}\right)\psi(x,y)}$ $= \langle xy|\frac{-i\varepsilon\hat{L}_z}{\hbar}|\psi\rangle$

$\Rightarrow \hat{L}_z$ is the generator of $\hat{R}_\phi$.

... which has the same matrix elements as if I defined $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$.

Transforming to polar coordinates, (see problem set),

$$\langle \rho\,\phi | \hat{L}_z | \psi \rangle = -i\hbar \frac{\partial}{\partial \phi} \psi(\rho, \phi).$$

Rotation operator $\hat{R}_\phi$ = finite rotations

$$\hat{R}_\phi = \left(\hat{R}_{\frac{\phi}{2}}\right)\left(\hat{R}_{\frac{\phi}{2}}\right) = \left(\hat{R}_{\frac{\phi}{N}}\right)^N$$ as with translation operator.

So ... if $N \to \infty$, $\frac{\phi}{N} \to \varepsilon$ infinitesimal

$$\hat{R}_\phi = \lim_{N\to\infty} \left(\hat{R}_{\frac{\phi}{N}}\right)^N$$

$$= \lim_{N\to\infty} \left(1 - \frac{i\phi}{N\hbar}\hat{L}_z\right)^N$$

$$= \exp\left(\frac{-i\phi}{\hbar}\hat{L}_z\right)$$