P4410 LECTURE 13

Derived infinitesimal translation operator $\hat{T}(\varepsilon) = \mathbb{1} - \frac{i\hat{p}\varepsilon}{\hbar} + \mathcal{O}(\varepsilon^2)$.

Now, consider its effect on operators — what Shankar calls a passive transformation: what is $\hat{T}^\dagger(\varepsilon)\hat{X}T(\varepsilon)$?

$$\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon) = \left(\mathbb{1} + \frac{i\varepsilon\hat{p}}{\hbar}\right)\hat{X}\left(\mathbb{1} - \frac{i\varepsilon\hat{p}}{\hbar}\right)$$

$$= \hat{X} + \frac{i\varepsilon\hat{p}}{\hbar}\hat{X} - \hat{X}\frac{i\varepsilon\hat{p}}{\hbar} + \mathcal{O}(\varepsilon^2)$$

$$= \hat{X} - \frac{i\varepsilon}{\hbar}[\hat{p},\hat{X}]$$

$$= \hat{X} + \varepsilon\mathbb{1}.$$

So $\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon)$ is an operator that measures "position + ε."

(or, as Shankar puts it, "position from an origin shifted by $-\varepsilon$"

And
$$\hat{T}^\dagger(\varepsilon)\hat{p}\hat{T}(\varepsilon) = \left(\mathbb{1} + \frac{i\varepsilon\hat{p}}{\hbar}\right)\hat{p}\left(\mathbb{1} - \frac{i\varepsilon\hat{p}}{\hbar}\right)$$

$$= \hat{p} + \frac{i\varepsilon\hat{p}^2}{\hbar} - \frac{i\varepsilon\hat{p}^2}{\hbar} + \mathcal{O}(\varepsilon^2)$$

$$= \hat{p}.$$

Now, what is $\hat{T}^\dagger(\varepsilon)\hat{H}\hat{T}(\varepsilon)$? Recall that for a unitary operator (such as $\hat{T}$), $\hat{U}^\dagger\hat{\Omega}(\hat{X},\hat{P})\hat{U} = \hat{\Omega}(\hat{U}^\dagger\hat{X}\hat{U}, \hat{U}^\dagger\hat{P}\hat{U})$.

(You showed on prob. set it's true for particle exchange operator – it's similar proof for any unitary operator) $\hookrightarrow$ see Shankar p. 286

So
$$\hat{T}^\dagger(\varepsilon)\hat{H}\hat{T}(\varepsilon) = \hat{H}\left(\hat{T}^\dagger\hat{X}\hat{T}, \hat{T}^\dagger\hat{P}\hat{T}\right)$$

$$= \hat{H}\left(\hat{X} + \varepsilon\mathbb{1}, \hat{P}\right)$$

So the classical-esque statement of invariance of the Hamiltonian can be restated in QM as $\hat{T}^\dagger(\varepsilon)\hat{H}\hat{T}(\varepsilon) = \hat{H}$.

Is momentum conserved?

$$\hat{T}^\dagger(\varepsilon)\hat{H}\hat{T}(\varepsilon) - \hat{H} = 0$$

$$= \left(\mathbb{1} + \frac{i\hat{p}\varepsilon}{\hbar}\right)\hat{H}\left(\mathbb{1} - \frac{i\hat{p}\varepsilon}{\hbar}\right) - \hat{H}$$

$$= \hat{H} + \frac{i\varepsilon\hat{p}\hat{H}}{\hbar} - \frac{i\varepsilon\hat{H}\hat{p}}{\hbar} - \hat{H}$$

$0 = -\frac{i\varepsilon}{\hbar}[\hat{p},\hat{H}]$ $\longleftarrow$ if $[\hat{A},\hat{H}]=0$ then $\hat{A}$ is conserved (assuming $\hat{H}$ is time-invariant).

So if $\hat{H}$ is invariant under a transformation, the generator of the infinitesimal translation is conserved (as in classical mechanics).

Finite transformations: How to build from infinitesimal.

$$\hat{T}(a) = \hat{T}\left(\tfrac{a}{2}\right)\hat{T}\left(\tfrac{a}{2}\right) = \left[\hat{T}\left(\tfrac{a}{N}\right)\right]^N$$

Now take $N \to \infty$, and $\frac{a}{N} \to \varepsilon$

$$T(a) = \lim_{N\to\infty}\left[\mathbb{1} - \frac{ia\hat{p}}{N\hbar} + \underbrace{\cancel{\mathcal{O}\left(\left(\tfrac{a}{N}\right)^2\right)}}_{0}\right]^N = e^{\frac{-ia\hat{p}}{\hbar}}$$

This trivially satisfies:

- Unitarity: $\hat{T}^\dagger\hat{T} = e^{ia\hat{p}/\hbar}\, e^{-ia\hat{p}/\hbar}$
- $\hat{T}(a+b) = \hat{T}(a)\hat{T}(b)$

~~Apply to position eigenstate: Matrix element:~~

$\cancel{\langle x'|\hat{T}(a)|x\rangle = \int dx''\, \langle x'|\hat{T}(a)|x''\rangle}$

Matrix element:

$$\langle x|\hat{T}(a)|x'\rangle = \langle x|e^{\frac{-i\hat{p}a}{\hbar}}|x'\rangle$$

$$= \int dp \langle x|e^{\frac{-i\hat{p}a}{\hbar}}|p\rangle\langle p|x'\rangle$$

$$= \int dp\, e^{\frac{-ipa}{\hbar}} \langle x|p\rangle\langle p|x'\rangle$$

$$= \int dp\, e^{\frac{-ipa}{\hbar}}\, e^{i(x-x')p/\hbar}$$

$$= \int dp\, e^{\frac{-ip}{\hbar}(x-x'-a)} \quad = \delta(x-x'-a)$$

and $\langle x|\hat{T}(a)|\psi\rangle = \int_{-\infty}^{\infty} dx' \langle x|\hat{T}(a)|x'\rangle\langle x'|\psi\rangle$

$$= \int_{-\infty}^{\infty} dx'\, \delta(x-x'-a)\,\psi(x')$$

$$= \psi(x-a)$$

can also show by: $\langle x|\hat{T}(a)|\psi\rangle = \int_{-\infty}^{\infty} dx' \langle x|e^{-\frac{i\hat{p}a}{\hbar}}|x'\rangle\langle x'|\psi\rangle$

$$= e^{\left(-a\frac{d}{dx}\right)}\psi(x)$$

taylor expand $\quad = \psi(x) + \psi'(x)(-a) + \frac{1}{2!}\psi''(x)(-a)^2 + \ldots$

$$= \psi(x'-a)$$

Time translation: The time translation operator takes $|\psi(t)\rangle$ and returns $|\psi(t+\varepsilon)\rangle$. This is just the propagator!

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$

where $i\hbar \frac{d}{dt}\hat{U}(t) = \hat{H}\hat{U}(t) \Longrightarrow \hat{U}(t) = e^{-i\hat{H}t/\hbar}$

so, in expansion, $\underbrace{\hat{U}_t(\varepsilon)}_{\text{also called } \hat{U}(t;t+\varepsilon)} = \mathbb{1} - \frac{i\varepsilon\hat{H}(t)}{\hbar} + \mathcal{O}(\varepsilon^2)$

$\Longrightarrow$ $\hat{H}$ itself is the generator of time translation.