P4410 LECTURE 12

Symmetries — a more sophisticated treatment.

Start with continuous symmetries in classical physics:

Consider an infinitesimal transformation:

$$q_i \to q_i + \delta q_i \quad \bar{q}_i, \quad \delta q_i \equiv \varepsilon \frac{\partial g}{\partial p_i}$$

$$p_i \to p_i + \delta p_i \quad \bar{p}_i, \quad \delta p_i \equiv -\varepsilon \frac{\partial g}{\partial q_i}$$

where q_i, p_i are generalized coordinates and momenta, and $g(q,p)$ is some (reasonably well-behaved) scalar function of the q's and p's. g is called the generator of the transformation.

Examine the case where $\mathcal{H}$ (classical Hamiltonian) is invariant under the transformation: $\delta\mathcal{H} = 0$.

$$\delta\mathcal{H} = \sum_i \frac{\partial\mathcal{H}}{\partial q_i}\left(\varepsilon\frac{\partial g}{\partial p_i}\right) + \frac{\partial\mathcal{H}}{\partial p_i}\left(-\varepsilon\frac{\partial g}{\partial q_i}\right) = 0$$

But the expression in the sum is a Poisson bracket:

$$\delta\mathcal{H} = \varepsilon\{\mathcal{H}, g\} = 0 \Rightarrow \{\mathcal{H}, g\} = 0$$

and g is a conserved quantity!

Now consider the case $g = p$ in 1 dimension $(q_i \to x)$

$$\delta x = \varepsilon\frac{\partial p}{\partial p} = \varepsilon$$

$$\delta p = -\varepsilon\frac{\partial p}{\partial x} = 0$$

So this is a translation: if the Hamiltonian is invariant under translations $(x \to x+\varepsilon)$ then momentum (the generator of translation) is conserved.

Now, the quantum analogue.

Define the infinitesimal translation such that $|\psi\rangle \to |\psi_\varepsilon\rangle$

gives $\langle\psi|\hat{x}|\psi\rangle \longrightarrow \langle\psi_\varepsilon|\hat{x}|\psi_\varepsilon\rangle = \langle\psi|x|\psi\rangle + \varepsilon.$

$\langle\psi|\hat{p}|\psi\rangle \longrightarrow \langle\psi_\varepsilon|\hat{p}|\psi_\varepsilon\rangle = \langle\psi|p|\psi\rangle$

since the classical limit should be obtained in the expectation value.

Define a translation operator $\hat{T}(\varepsilon)$: $|\psi_\varepsilon\rangle = \hat{T}(\varepsilon)|\psi\rangle$

so
$$\langle\psi|\hat{T}^\dagger(\varepsilon)\hat{x}\hat{T}(\varepsilon)|\psi\rangle = \langle\psi|x|\psi\rangle + \varepsilon$$
$$\langle\psi|\hat{T}^\dagger(\varepsilon)\hat{p}\hat{T}(\varepsilon)|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle$$

But since this must be true for $\underline{\text{any}}$ $|\psi\rangle$, we can write
$$\hat{T}^\dagger(\varepsilon)\hat{x}\,\hat{T}(\varepsilon) = \hat{x} + \varepsilon \qquad (\text{really } \hat{x} + \varepsilon\hat{I})$$
$$\hat{T}^\dagger(\varepsilon)\hat{p}\,\hat{T}(\varepsilon) = \hat{p}$$

On, a position eigenstate, $\hat{T}$ must act as:
$$\hat{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle$$

so for any $|\psi\rangle$,
$$\rightarrow \quad |\psi_\varepsilon\rangle = \hat{T}|\psi\rangle \qquad \text{use completeness relation}$$
$$= \int_{-\infty}^{\infty} dx'\ \hat{T}(\varepsilon)|x'\rangle\langle x'|\psi\rangle$$
$$= \int dx'\ |x'+\varepsilon\rangle\langle x'|\psi\rangle$$

So $\hat{T}(\varepsilon) = \int dx'\ |x'+\varepsilon\rangle\langle x'|$;

matrix element $T(\varepsilon)_{xx'} = \langle x|\left(\int dx''\,|x''+\varepsilon\rangle\langle x''|\right)|x'\rangle$
$$= \int dx''\ \delta(x - x'' - \varepsilon)\ \delta(x'' - x')$$

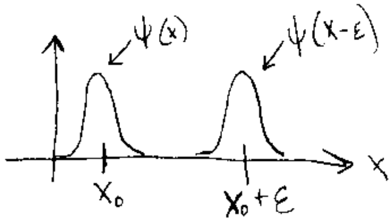
$$T(\varepsilon)_{xx'} = \delta(x - x' - \varepsilon)$$

What is wave function of transformed state? $\langle x|\psi_\varepsilon\rangle$

$$=\langle x|\hat{T}(\varepsilon)|\psi\rangle = \int dx'\, T(\varepsilon)_{xx'} \langle x'|\psi\rangle = \int dx'\, \delta(x - x' - \varepsilon)\langle x'|\psi\rangle$$

$$= \langle x' - \varepsilon|\psi\rangle$$

so $\psi_\varepsilon(x) = \psi(x-\varepsilon)$ as expected.



So we haven't yet used our behavior-under-$\hat{p}$ condition, while defining $\hat{T}$ completely. So now let's check that it's satisfied

$$\langle\psi_\varepsilon|\hat{p}|\psi_\varepsilon\rangle = \int dx' \int dx'' \langle\psi_\varepsilon|x'\rangle\langle x'|\hat{p}|x''\rangle\langle x''|\psi_\varepsilon\rangle$$

$$= \int dx' \int dx''\, \psi_\varepsilon^*(x')\left[-i\hbar\,\delta(x'-x'')\frac{d}{dx''}\right]\psi_\varepsilon(x'')$$

$$= -i\hbar \int dx' \int dx''\, \psi^*(x'-\varepsilon)\,\delta(x'-x'')\frac{d}{dx''}\psi(x''-\varepsilon)$$

$$= -i\hbar \int dx''\, \psi^*(x''-\varepsilon)\frac{d}{dx''}\psi(x''-\varepsilon)$$

But since integral is from $-\infty$ to ∞, we can substitute variable $x'' \to x'' + \varepsilon$

$$= -i\hbar \int dx''\, \psi^*(x'')\frac{d}{dx''}\psi(x'') = \langle\psi|\hat{p}|\psi\rangle \text{ as expected.}$$

Now, define a translation invariant Hamiltonian as one where:

$$\langle \psi_\varepsilon | \hat{H} | \psi_\varepsilon \rangle = \langle \psi | \hat{H} | \psi \rangle \quad \text{for any } |\psi\rangle.$$

To proceed, investigate $\hat{T}$ further: treat it as an operator expansion.

$$\hat{T}(\varepsilon = 0) = \mathbb{1}.$$

For small ε: expand about $\varepsilon = 0$

$$\hat{T}(\varepsilon) = \mathbb{1} + \hat{\alpha}\varepsilon + \cancel{\mathcal{O}(\varepsilon^2)} + \ldots$$ ignore higher order terms. Let $\hat{\alpha} \equiv \frac{-i\hat{G}}{\hbar}$

So for small ε, $\hat{T}(\varepsilon) = \mathbb{1} - \frac{i\hat{G}}{\hbar}\varepsilon$

Require $\langle \psi_\varepsilon | \psi_\varepsilon \rangle = \langle \psi | \psi \rangle$:

$$\Longrightarrow \hat{T}^\dagger(\varepsilon)\hat{T}(\varepsilon) = \mathbb{1} + \mathcal{O}(\varepsilon^2)$$

$$\left(\mathbb{1} + \frac{i\hat{G}^\dagger}{\hbar}\varepsilon\right)\left(\mathbb{1} - \frac{i\hat{G}}{\hbar}\varepsilon\right) = \mathbb{1} + \mathcal{O}(\varepsilon^2) + \frac{i\varepsilon}{\hbar}\left(\hat{G}^\dagger - \hat{G}\right)$$

$$\Rightarrow \hat{G}^\dagger - \hat{G} = 0 \rightarrow \hat{G} \text{ is Hermitian.}$$

OK, now return to $\langle x | \hat{T}(\varepsilon) | \psi \rangle = \psi(x - \varepsilon)$

Expand in ε: $\quad = \psi(x) - \varepsilon \frac{d}{dx}\psi(x) + \mathcal{O}(\varepsilon^2)$

$$\langle x | \mathbb{1} - \frac{i\varepsilon\hat{G}}{\hbar} | \psi \rangle + \mathcal{O}(\varepsilon^2) = \psi(x) \frac{-i\varepsilon}{\hbar} \langle x | \hat{G} | \psi \rangle + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \frac{-i}{\hbar} \langle x | \hat{G} | \psi \rangle = -\frac{d}{dx} \langle x | \psi \rangle$$

or $\langle x' | G | x \rangle = -i\hbar \frac{d}{dx'} \langle x' | x \rangle = -i\hbar \frac{d}{dx} \delta(x'-x)$

$$= -i\hbar\, \delta(x'-x) \frac{d}{dx}$$

$$\Longrightarrow \hat{G} = \hat{p}.$$ momentum operator is generator of translation: $\hat{T}(\varepsilon) = \mathbb{1} - \frac{i\varepsilon\hat{p}}{\hbar} + \mathcal{O}(\varepsilon^2)$.