P4410 LECTURE 10

The Levi-Civita "tensor" (not really a tensor) — an antisymmetric shorthand.

In 3 indices: $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$

all other $\varepsilon_{ijk} = 0$

In >3 indices: $\varepsilon_{ijkl\ldots} = 0$ if any two indices the same (i.e. $\varepsilon_{1223} = 0$)

$\varepsilon_{ijkl\ldots} = -1$ for odd permutations

$\varepsilon_{ijkl\ldots} = 1$ for even permutations: a permutation is even if it can be arrived at by reversing pairs of indices an even # of times:

$$\underset{\text{even}}{1234} \to \underset{\text{odd}}{1243} \to \underset{\text{even}}{1423} \to \underset{\text{odd}}{3421} \to \underset{\text{even}}{4321}$$

The Density operator — mixed ensembles.

So far we have (sloppily) referred to "mixed states," meaning a system that's not in an eigenstate of whatever operator we're considering. We talked of expectation value $\langle\psi|\hat{A}|\psi\rangle = \langle A\rangle$ as the average value of A measured on an ensemble of systems in state $|\psi\rangle$, which could be a superposition of eigenstates of some operator. Key is all those systems are in the same state $|\psi\rangle$ — i.e. same vector in Hilbert space. This is a coherent or pure ensemble. Now we have to introduce the concept of an ensemble of systems which might not all be in the same state.

Introduce the weighting factor / probability / fractional population w_i
→ so that if we have N particles, $w_i N$ of them are in state $|i\rangle$. In this case, the $|i\rangle$ are not necessarily basis states; they don't even have to be orthogonal! (if I have a particle in state $|i\rangle$ there's no reason a second one has to be in an orthogonal state.)
Example: a system of spin-$\frac{1}{2}$ particles. Work in the $\hat{S}_z$ basis.

A coherent ensemble: N particles, all in state $(|+\rangle + |-\rangle)/\sqrt{2}$.
An incoherent ensemble: $\frac{N}{2}$ particles in $|+\rangle$, $\frac{N}{2}$ in $|-\rangle$. $w_+ = \frac{1}{2}$, $w_- = \frac{1}{2}$

Are these ensembles distinguishable? Take ensemble average $\langle \bar{S}_z \rangle$ — this is sort of an average of expectation values:

$$\langle \bar{S}_z \rangle = \sum_i w_i \langle i | \hat{S}_z | i \rangle$$

Coherent case: call $|a\rangle \equiv \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. $w_a = 1$, all other $w_i = 0$

so $\langle \bar{S}_z \rangle = \langle a | \hat{S}_z | a \rangle = \frac{1}{2} \frac{\hbar}{2} (1 \;\; 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$

Incoherent case: $\langle \bar{S}_z \rangle = \frac{1}{2} \langle + | \hat{S}_z | + \rangle + \frac{1}{2} \langle - | \hat{S}_z | - \rangle$

$$= \frac{1}{2}\left(\frac{\hbar}{2}\right) + \frac{1}{2}\left(\frac{-\hbar}{2}\right) = 0. \quad \rightarrow \text{same result.}$$

But S_z isn't the only thing we could have measured. Try $\hat{S}_x \Leftrightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Coherent case: $\langle \bar{S}_x \rangle = \langle a | \hat{S}_x | a \rangle = \frac{1}{2} \frac{\hbar}{2} (1 \;\; 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}$

Incoherent case: $\langle \bar{S}_x \rangle = \frac{1}{2} \langle + | \hat{S}_x | + \rangle + \frac{1}{2} \langle - | \hat{S}_x | - \rangle$

$$= \frac{1}{2} \left[\frac{\hbar}{2} (1 \;\; 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\hbar}{2} (0 \;\; 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 0.$$

So these are clearly very different ensembles! We can quantify all the information available about a mixed ensemble in the <u>density matrix</u> (John von Neumann).

Start from the ensemble average of a quantity A: First, note that $\sum_i w_i = 1$. And that sum does <u>not</u> have to be over the dimension N of the vector space — can be <u>much</u> more, since in theory <u>any</u> ket in Hilbert space can have a nonzero population in the ensemble.

$$\langle \bar{A} \rangle = \sum_i w_i \langle i | \hat{A} | i \rangle \qquad \longleftarrow \sum \text{ over states in ensemble}$$

Now, expand $|i\rangle$ into basis states $|\alpha_j\rangle$.

$$= \sum_i w_i \langle i | \left(\sum_{j=1}^{N} |\alpha_j\rangle\langle\alpha_j| \right) \hat{A} \left(\sum_{k=1}^{N} |\alpha_k\rangle\langle\alpha_k| \right) |i\rangle \qquad \longleftarrow \; i \; \sum \text{ over states in } \underline{\text{ensemble}}$$

j,k $\sum$ over states in <u>basis</u>

$$= \sum_i w_i \sum_{j,k=1}^{N} \langle i|\alpha_j\rangle\langle\alpha_j|\hat{A}|\alpha_k\rangle\langle\alpha_k|i\rangle$$

$$= \sum_{j,k=1}^{N} \underbrace{\left(\sum_i w_i \langle\alpha_k|i\rangle\langle i|\alpha_j\rangle \right)}_{\equiv \rho_{kj}} A_{jk}$$

These are elements of the density operator

$$\hat{\rho} \equiv \sum_i w_i |i\rangle\langle i|$$

Rewrite ensemble average $\langle \bar{A} \rangle = \sum_{j,k=1}^{N} \rho_{kj} A_{jk} = \mathrm{Tr}(\hat{\rho}\hat{A})$.