P4410 LECTURE 1/2

Goal of this class is to (first) firm up the mathematical concepts introduced in P3220, and (then) push into new physics techniques and applications. Shankar will be the primary text, with some topics out of Liboff (roughly the reverse emphasis ratio as P3220).

Quantum system is in a <u>state</u>, represented by a <u>ket</u> $|\alpha\rangle$.

- α is a <u>label</u> identifying the state. We have to pick a labeling scheme appropriate to the problem and the system at hand —— so if I refer to the state $|17\rangle$ that's meaningless unless I tell you also that, say, we're dealing with an infinite 1-dimensional square well and $|n\rangle$ is the n'th energy eigenstate. Ket labeling schemes are [redacted] generally tied to the <u>basis</u> (or representation) we're working in.
- The set of all possible kets (states) for a given system forms a <u>linear vector space</u>. This is a much more general concept than spatial "arrow" vectors. Properties of a linear vector space $\mathbb{V} = \{|A\rangle, |B\rangle, |C\rangle, |D\rangle, \ldots\}$ (Shankar p. 2)

  - Vectors can be added
  - Vectors can be multiplied by scalars
  - Both operations result in another element of $\mathbb{V}$:
    $|A\rangle + |B\rangle \in \mathbb{V}$, $a|A\rangle \in \mathbb{V}$.
  - Scalar multiplication is <u>distributive</u> in vectors <u>and</u> scalars:
    $a(|A\rangle + |B\rangle) = a|A\rangle + a|B\rangle$, $(a+b)|A\rangle = a|A\rangle + b|A\rangle$.

- Addition is commutative: $|A\rangle + |B\rangle = |B\rangle + |A\rangle$
  and associative: $|A\rangle + (|B\rangle + |C\rangle) = (|A\rangle + |B\rangle) + |C\rangle$

- A <u>null vector</u> $|0\rangle$ (often just written 0) exists with properties:

$$0|A\rangle = |0\rangle$$
$$|A\rangle + |0\rangle = |A\rangle$$

- Every vector has an inverse under addition: $|-A\rangle = -1|A\rangle$ such that $|A\rangle + |-A\rangle = |0\rangle$.

If the scalars a, b mentioned above are all real, we have a <u>real vector space</u>. In QM we are usually concerned with <u>complex</u> vector spaces (i.e. the scalars may be complex).

- A vector space has dimension n if it contains a maximum of n linearly <u>independent</u> vectors. A set of vectors is linearly <u>dependent</u> if $|i\rangle = \sum_{j \neq i} C_j |j\rangle$ for any $|i\rangle$ in the set.

The most common vector space dimensions in QM are probably 2 and ∞.

Note that this definition of a vector space is <u>much</u> more inclusive than that of a spatial "arrow" vector with a length and direction.

- For an n-dimensional vector space, any set of n linearly independent vectors forms a basis that spans the vector space. If we call the basis vectors $|i\rangle$, then any vector in $\mathbb{V}$ can be written $|A\rangle = \sum_{i=1}^{n} c_i |i\rangle$.

A quick proof:

- Assume $|A\rangle \neq \sum_{i=1}^{n} c_i |i\rangle$, where the $|i\rangle$ are linearly independent, for any set of coefficients c_i.
- Following our definition, this means $|A\rangle$ is linearly independent of all the $|i\rangle$.

$\Rightarrow$ The dimension of the vector space must be $\geq n+1$.

A basis provides a way to describe vectors and perform operations on them, using the components c_i:

$$\text{Let } |A\rangle = \sum_i a_i |i\rangle, \quad |B\rangle = \sum_i b_i |i\rangle$$

$$|A\rangle + |B\rangle = \left(\sum_i a_i |i\rangle\right) + \left(\sum_i b_i |i\rangle\right) = \sum_i (a_i + b_i) |i\rangle$$

so the components of $|A\rangle + |B\rangle$ are the sums of the components of $|A\rangle$ and $|B\rangle$. Pretty obvious — confirms that components of linear vector space elements behave as you would expect based on "arrow" vectors you can visualize.

→ A notational annoyance: Scalar multiplied by vector.
$a|A\rangle$ is often written $|aA\rangle$ — not strictly kosher since A is only a label and no one ever said you could multiply a scalar and a label! (Often it can get you in big trouble!) But this usage is common in Shankar and Liboff — just be careful to understand what it means.

<u>Inner products</u> — Think of this as a generalization of the dot product. Denote the inner product of two vectors $|A\rangle$ and $|B\rangle$ as $\langle A|B\rangle$. For a complex vector space, we require that $\langle A|B\rangle = \langle B|A\rangle^*$.

$\longrightarrow$ This implies $\langle A|A\rangle$ is real. To define the "length" (norm) of the vector as $\sqrt{\langle A|A\rangle}$, need to require $\langle A|A\rangle \geq 0$ too.

Also, linearity: $\langle A|\big(b|B\rangle + c|C\rangle\big) = b\langle A|B\rangle + c\langle A|C\rangle$.

Reversing order must bring in a complex conjugate, so the inner product of $b|B\rangle + c|C\rangle$ and $|A\rangle$ must be $b^*\langle B|A\rangle + c^*\langle C|A\rangle$.

More properties:

- Two vectors $|A\rangle, |B\rangle$ are <u>orthogonal</u> if $\langle A|B\rangle = 0$.
- A vector is <u>normalized</u> if $\langle A|A\rangle = 1$.
- A basis of vectors that are mutually orthogonal and all have norm 1 is an <u>orthonormal</u> <u>basis</u>: $\langle i|j\rangle = \delta_{ij}$.

Inner product via components: Let $|A\rangle = \sum_{i=1}^{n} a_i|i\rangle$, $|B\rangle = \sum_{i=1}^{n} b_i|i\rangle$.

Now, $\langle A|B\rangle = \sum_{i,j} a_i^* b_j \langle i|j\rangle$

If basis is orthonormal: $\langle A|B\rangle = \sum_{i,j} a_i^* b_j \delta_{ij} = \sum_i a_i^* b_i$.

Equivalent to matrix mult. of row & column: $\begin{pmatrix} a_1^* & a_2^* & a_3^* & \cdots & a_n^* \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$

Consider the column vector the basis (component) representation of ket $|B\rangle$; the row vector of complex conjugates of components is the representation of bra $\langle A|$, also called the adjoint of $|A\rangle$.

Technically, the bra vectors form a distinct vector space that is dual to the kets. (We don't have much use for this detail - just remember the rows vs. columns & complex conjugation if this is confusing).

Basis operations:

If $|1\rangle \cdots |n\rangle$ form an orthonormal basis, what is the i^{th} component of vector $|A\rangle$?

$$|A\rangle = \sum_{j=1}^{n} a_j |j\rangle$$

operate from left with $\langle i|$:

$$\langle i|A\rangle = \langle i| \left(\sum_{j=1}^{n} a_j |j\rangle \right)$$

$$= \sum_{j=1}^{n} \langle i| a_j |j\rangle$$

$$= \sum_{j=1}^{n} a_j \langle i|j\rangle$$

orthonormal:
$$= \sum_{j=1}^{n} a_j \delta_{ij}$$

$$= a_i$$

So $a_i = \langle i|A\rangle$.

From this we can also get another very useful identity:

$$|A\rangle = \sum_{j=1}^{n} a_j|j\rangle \quad \text{replace } a_j \text{ with } \langle j|A\rangle:$$

$$= \sum_{j=1}^{n} \underbrace{\langle j|A\rangle}_{\substack{\text{This is a} \\ \text{scalar} \rightarrow \text{move it to right}}} |j\rangle$$

$$|A\rangle = \sum_{j=1}^{n} |j\rangle\langle j|A\rangle$$

Since this is true for any $|A\rangle \in \mathbb{V}$, we conclude that $\sum_{j=1}^{n} |j\rangle\langle j| = \mathbb{1}$, the identity operator. This identity is often called the completeness relation and it's only a property of an orthonormal basis.

(Less trivial) operators: Operator acts on a ket to produce a new ket: $\hat{\Omega}|A\rangle = |A'\rangle$ (Shankar omits "hat" notation)

Similarly, the Hermitian adjoint of an operator $\hat{\Omega}^\dagger$ is defined such that the bra operating on the operator yields the adjoint bra to the new ket:

$$\langle A|\hat{\Omega}^\dagger = \langle A'| .$$

Often referred to (slightly misleadingly) as the operator $\hat{\Omega}^\dagger$ "operating to the left."

An operator is <u>linear</u> if $\hat{\Omega}\big(|A\rangle + |B\rangle\big) = \hat{\Omega}|A\rangle + \hat{\Omega}|B\rangle$.

Can be represented by a matrix in an orthonormal basis:

Assume the effect of $\hat{\Omega}$ on basis states $|i\rangle$ is "known:"

$$\hat{\Omega}|i\rangle = |i'\rangle, \quad \text{whose components are } \langle j|i'\rangle.$$

NOT nec. a basis state!

$$\langle j|i'\rangle = \langle j|\hat{\Omega}|i\rangle \equiv \Omega_{ji}$$

<u>matrix elements of Ω</u>

Note that these matrix elements are <u>basis-specific</u>.

For a generic vector $|A\rangle = \sum_i a_i |i\rangle$, let $\hat{\Omega}|A\rangle = |A'\rangle = \sum_i a_i' |i\rangle$

In matrix form:

$$\begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2n} \\ \vdots & \vdots & & \vdots \\ \Omega_{n1} & \Omega_{n2} & \cdots & \Omega_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

It's easy to show that $\Omega^{\dagger}_{ij} = \Omega^{*}_{ji}$.

Example: $\mathbb{1} \Rightarrow \quad \mathbb{1}_{ij} = \langle i|\mathbb{1}|j\rangle = \langle i|j\rangle = \delta_{ij}$

so $\mathbb{1} \leftrightarrow \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & 1 & \\ & & \ddots \; 1 \end{pmatrix}$

"is represented as"

Operator products: in general, $\hat{\Omega}\hat{\Lambda} \neq \hat{\Lambda}\hat{\Omega}$.

$$\hat{\Omega}\hat{\Lambda} - \hat{\Lambda}\hat{\Omega} \equiv [\hat{\Omega}, \hat{\Lambda}] \quad \text{commutator}$$

$$\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega} \equiv \{\hat{\Omega}, \hat{\Lambda}\} \quad \text{anticommutator}$$

Projection operator: $\hat{P}_i = |i\rangle\langle i|$ projects out the i^{th} component of an arbitrary ket:

$$|A\rangle = \sum_j a_j |j\rangle$$

$$\begin{aligned}
\hat{P}_i|A\rangle &= |i\rangle\langle i| \sum_j a_j |j\rangle \\
&= |i\rangle \sum_j a_j \langle i|j\rangle \\
&= |i\rangle \sum_j a_j \delta_{ij} \\
&= a_i |i\rangle .
\end{aligned}$$

Matrix elements?

$$|i\rangle\langle i| \leftrightarrow \begin{array}{c l} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{c} i^{th}\text{ column} \\ \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix} \end{array} \\ i^{th}\text{ row} & \end{array}$$

$$= \begin{array}{c l} \begin{array}{c} i^{th}\text{ column} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} & \begin{array}{l} \\ \\ \\ \\ \leftarrow i^{th}\text{ row} \\ \\ \\ \end{array} \end{array}$$

Verify that this projection operator has the above properties!