Filters and Waveform Shaping

Purpose
The aim of this experiment is to study the frequency filtering properties of passive (R, C, and L) circuits for sine waves, and the transient response of the same circuits to square waves. Filters are important in experiments for enhancing signals of particular interest while suppressing unwanted background.

The oscilloscope probe is introduced in this lab.

Introduction
A frequent problem in physical experiments is to detect an electronic signal when it is hidden in a background of noise and unwanted signals. The signal of interest may be at a particular frequency, as in an NMR experiment, or it may be an electrical pulse, as from a nuclear particle detector. The background generally contains thermal noise from the transducer and amplifier, 60 Hz power pick up, transients from machinery, radiation from radio and TV stations, cell phone radiation, and so forth. The purpose of filtering is to enhance the signal of interest by recognizing its characteristic time dependence and to reduce the unwanted background to the lowest possible level.

In everyday life, your radio does this when you tune to a particular station, using a resonant circuit to recognize the characteristic frequency. The signal you want may be less than $10^{-6}$ of the total radiation power at your antenna, yet you get a high quality signal from the selected station.

A filter freely transmits electrical signals within a certain range of frequencies called the pass band, and suppresses signals at all other frequencies (the attenuation bands). The boundary between a pass band and an attenuation band is called the cut-off frequency $f_c$. We usually define $f_c$ to be the frequency at the half-power point or 3dB point, where the power transmitted is half the maximum power transmitted in the pass band. The output voltage amplitude at $f = f_c$ is $1 / \sqrt{2} = 70.7\%$ of the maximum amplitude.

There are three basic types of filter. The high-pass filter transmits signals at frequencies above the cut-off $f_c$ and blocks them at lower frequencies. The low-pass filter transmits signals below $f_c$ and blocks higher frequencies. The band-pass filter, often taking the form of a resonant circuit, transmits a certain band of frequencies and blocks signals outside that band. For band-pass filters the bandwidth is the range of frequencies between the upper ($f_+$) and lower ($f_-$) half power points: $\text{bandwidth} = \Delta f = f_+ - f_-.$

Many experiments require specific filters designed so that the signal from the phenomenon of interest lies in the pass-band of the filter, while the attenuation bands are chosen to suppress the background and noise.

This experiment introduces you to the filtering properties of some widely used but simple circuits, employing only a resistor and capacitor for high- and low-pass filters and an LCR circuit for band-pass.
Readings

1. FC Sections 3.4 – 3.18 and 10.1 – 10.6
2. (optional) H&H Chapter 1, especially sections 1.13-1.24. You will make frequent use of the last topic in Section 1.18, “Voltage Dividers Generalized.” Appendix A on oscilloscope probes.
3. (optional) D.V. Bugg, Chapter 14. The theory of series and parallel LCR circuits is presented in detail.

Theory

RC FILTERS
The response of RC low-pass and high-pass filters to sine waves is discussed in FC Sections 3.9&3.10. The most important formula for both cases is

\[ f_c = \frac{1}{2\pi RC} \]

where \( f_c \) is the 3 dB or half-power point. The response of RC circuits to square waves is in FC Sections 10.1 – 10.4 (H&H Sections 1.13 and 1.14). The main result is that the exponential rise and decay times \( t_R \) and \( t_D \) are equal to \( RC \).

LCR RESONANT CIRCUIT
See FC Section 3.12 (H&H Section 1.22). The resonant frequency and Q are given by

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \quad Q = \omega_0 RC \]

where \( \omega_0 = 2\pi f_0 \).

The transient or step response of an LCR circuit is covered in D.V. Bugg, Chapter 14. The response is similar to that of the damped harmonic oscillator from mechanics. Bugg derives a differential equation for the current \( I_1 \) through the inductor (Bugg Eqn. 14.31):

\[ \frac{V}{CLR} = \frac{d^2 I_1}{dt^2} + \frac{1}{CR} \frac{dI_1}{dt} + \omega_0^2 I_1 \]

where \( V \) is the input voltage. The general solution to this equation is

\[ I_1 = e^{-\gamma t} \left( A \cos \omega_1 t + B \sin \omega_1 t \right) + \frac{V}{R} \]

where

\[ \gamma = \frac{1}{2RC}, \quad \omega_1 = 2\pi f_1 = \sqrt{\omega_0^2 - \gamma^2}, \quad \omega_0 = 2\pi f_0. \]

At the rising voltage step, the boundary conditions are \( I_1(0) = 0 \) and \( dI_1/dt = 0 \). This leads to the solutions
\[ A = -\frac{V}{R}, \quad B = -\frac{V}{2\omega RC}. \]

Our measured \( V_{\text{out}} \) is the voltage across the inductor, which is given by

\[ V_{\text{out}} = L \frac{dI_1}{dt}. \]

Putting everything together we find the result

\[ V_{\text{out}} = V_{\text{in}} \frac{1}{\omega_1 RC} e^{-\gamma t} \sin(\omega_1 t) \]

The response is a sine-wave with an exponentially decaying envelope or amplitude. The exponential decay time of the envelope is \( 1/\gamma \). The natural frequency \( f_1 \) can be found by measuring the time \( t_1 \) between zero-crossings, and then \( f_1 = 1/t_1 \).

Problems

1. (A) Calculate the cut-off frequency \( f_\text{c} \) in kHz for the low-pass filter in Figure 3.1 (a). Draw a Bode plot of the frequency response, sketching in the correct form near the half power point. (A Bode plot is a log-log plot of \( |V_{\text{out}}/V_{\text{in}}| \) versus frequency. See H&H Fig. 4.31 for an example. Note that the axes are labeled in decades and the x and y step sizes are equal, so a \( 1/f \) frequency response appears as a line with a slope of \(-1\).)

   (B) Calculate the integrating time constant \( t_\text{R} \). Draw the output waveform where the input is a low-frequency square wave. Low frequency in this case means that the period of the wave is much longer than the RC time constant.

2. (A) Calculate the cut-off frequency for the high-pass filter in Figure 3.1 (b) and draw its Bode plot.

   (B) Calculate the differentiating time constant \( t_\text{D} \) and draw the output waveform for input square waves.

3. Derive the expression for the resonant frequency of the resonant circuit in Figure 3.2.

4. For a resonant circuit the characteristic impedance \( Z_0 \) is the magnitude of the impedance of the inductor or the capacitor at the resonant frequency:

\[ Z_0 = \omega_0 L = \frac{1}{\omega_0 C} = \frac{\sqrt{L}}{\sqrt{C}} \]
Calculate the resonant frequency $f_0$, the characteristic impedance $Z_0$, and the quality factor $Q$ for the band-pass filter in Figure 3.2. Make a Bode plot showing the expected attenuation versus frequency and mark on it the two half power points $f_+$ and $f_-$. This plot is a bit harder to make since it is not just straight lines. You can plot some points near the resonance by hand or use a computer plotting program.

Figure 3.1

Experiment #3  3.4
The Experiment

Make up the following circuits on your board (Figure 3.4):

a) Resistive divider.
   \[ R = 10 \, \text{k} \Omega \]
   \[ R = 6.8 \, \text{k} \Omega \]

b) Low-pass filter
   \[ R = 10 \, \text{k} \Omega \]
   \[ C = 1000 \, \text{pF} \]

c) High-pass filter
   \[ R = 10 \, \text{k} \Omega \]
   \[ C = 1000 \, \text{pF} \]

d) Bandpass filter
   \[ R = 10 \, \text{k} \Omega \]
   \[ C = .01 \, \mu\text{F} \]
   \[ L = 10 \, \text{mH} \]
If you cannot find components in stock with the specified values, take the nearest in value that you can find, within 30% if possible.

1A. Measure the value of each resistor and capacitor before you insert them. Why before?

1B. Calculate the expected attenuation of the divider. Calculate the expected values of the cut-off frequencies for the high- and low-pass filters using the actual component values. Calculate the expected resonant frequency $f_0$ and quality factor $Q$ for the band-pass filter using the actual component values.

1C. Sketch approximate Bode diagrams for each circuit. Make the axes large and clear so you can plot data on these same graphs as you observe the actual behavior of the circuits below. The frequency range should cover at least $f = 10^{-3} f_c$ to $f = 10^3 f_c$.

TEST SET-UP
Connect the circuit board to the function generator, the oscilloscope and the counter/timer as shown in Figure 3.5.
• Set the oscilloscope to display CH 1 and CH 2 using sweep A triggered by CH 4. Use dc coupling.
• Test the system with 1 kHz sine waves at 1 volt p-p. Display CH 1 and CH 2 waveform on the screen.

MEASUREMENTS WITH SINE WAVES
First make connections to the voltage divider. Vary the frequency over a wide range with the function generator. Verify that the attenuation ($=V_{out}/V_{in}$) is independent of frequency up to a very high frequency. Compare the measured value of attenuation with that predicted from component values. In the questions below, you do not need to do an error analysis. We do expect that you will notice when your measurement is very different from what is expected and will correct your mistakes.
2A. What is the phase difference (in degrees) between $V_{in}$ and $V_{out}$? Is the value what you expect? (See Fig. 1.60 of H&H for an example of a phase plot.)

2B. If there is a high frequency cut-off, measure its value (where the voltage is reduced to 0.7 of the output value) and explain the cut-off quantitatively. Consider the capacitance of the cable (about 10 pF per foot) and the scope input.

2C. Repeat the measurement using the 10x probe to observe the output of the circuit. Explain your observation. Note that the oscilloscope knows when the 10X probe is attached and automatically adjusts the scales to give the correct value.

Next study the high-pass and low-pass filters.

3A. Determine the frequency at the half power point $f_c$ for each filter. Compare your measured half power point ($V_{out}/V_{in} = 0.707$) with the cut-off frequency computed from the actual component values used, rather than the nominal values in the prelab.

3B. Measure the attenuation vs. frequency at decade intervals from $f = 10^{-3} f_c$ to $f = 10^3 f_c$ if possible. Test the predicted frequency response by marking your data points directly on your two Bode plots.

3C. Measure the phase shift between input and output sine waves at $f = 0.1 f_c$, $f_c$, and $10 f_c$. Plot these data on a phase diagram. (See H&H Fig. 1.60 for an example of a phase diagram.)

Use the 10X probe for the remaining measurements because it is less of a perturbation to your circuits.
Finally look at the band-pass filter. Find the resonant pass frequency $f_0$ two ways.

4A. Adjust the frequency so that:
   a) The output has maximum amplitude ($V_{out}/V_{in} = \text{max}$).
   b) There is zero phase difference between $V_{out}$ and $V_{in}$.

4B. Which method is more precise?

4C. Determine the quality factor $Q$ by measuring the frequencies at the two half-power points $f_+$ and $f_-$ above and below the resonance at $f_0$. The quality factor is defined by

$$Q = \frac{\text{Resonant frequency} f_0}{\text{Bandwidth} \Delta f} \quad \text{where} \quad \Delta f = f_+ - f_- .$$

4D. Map out the shape of the resonance curve by measuring the attenuation at $f_0$, $f_0 \pm \Delta f/4$, $\pm \Delta f/2$, $\pm \Delta f$, $\pm 2\Delta f$, $\pm 5\Delta f$.

4E. Compare the measured $f_0$ with the expected value $1/(2\pi \sqrt{LC})$. Calculate a corrected value of $L$ from the measured values of $f_0$ and $C$, and use this value in the future. Compare this value of $L$ to the one you found using the meter in the lab.

4F. Compare the measured value of $Q$ with that predicted from measurements of component values. It is common in all electrical circuits to find $Q$ values that are somewhat lower than values you predict using measured component values. This is due to additional losses in the circuit, in this case probably losses in the inductor. If you really want to understand this in detail, you have to find a way to measure the inductance $L$ and the inductor’s “equivalent series resistance” (ESR) at the frequency of interest, and then come up with an equation that shows how the ESR reduces the $Q$. As a first approximation, the ESR is simply the resistance of the wire in the inductor that you can measure with the DMM.

4G. Test the predicted frequency response by plotting your data on your predicted resonance curve. (This last step should be done using a computer to plot the predicted frequency response based on your measured values for $R$, $L$, and $C$.)

**MEASUREMENTS WITH SQUARE WAVES**

Examine the transient response excited by the voltage step at the beginning of each square wave. Transient responses indicate how each circuit will affect information in the form of pulses such as those produced by particle and photon detectors. When used to process pulses:

- A **low-pass filter** is called an integrator circuit.
- A **high-pass filter** is called a differentiator circuit.
- A **band-pass filter** is called a damped oscillator.
Change the function generator output to square waves 1 V p-p. Display waveforms for $V_{in}$ and $V_{out}$.

- Reduce the frequency until the interval between steps is long enough to allow transients to decay within 1% of final voltage before the next step.
- Use cursors for accurate measurement of the quantities $t_R$, $t_D$, $\gamma$, and $f_1$ (see below).
- Example—to measure $t_R$ for integrator:
  a) Measure $V_{in}$ of input wave.
  b) Set voltage cursor at 63% $V_{in}$.
  c) Identify intersection of cursor with wave.
  d) Use time cursors to measure interval from start of transient to intersection at 63%.

For the divider, integrator, and differentiator circuits (see Fig. 3.6):

**5A.** Measure exponential rise-time $t_R$ for the divider and integrator. Measure the decay-time $t_D$ for the differentiator.

**5B.** Do the results satisfy the expected relations?

Note that:

a) $t_R = RC$ for the integrator, and $t_D = RC$ for the differentiator.

b) $f_c$ (low pass) = $(2\pi t_R)^{-1}$ and $f_c$ (high pass) = $(2\pi t_D)^{-1}$ (Where the cut-off frequencies $f_c$ are the values measured from attenuation of the sine waves.)

For the resonant band-pass filter:

- Observe the waveforms and check that the transient has decayed to less than 1% between the steps of the square wave. Lower the frequency if necessary.

**6A.** Measure the natural resonance frequency $f_1$ using the time cursors. Take your measurements using baseline crossings rather than at peaks (why?). If you measure the period $t_1$ (the time between every other zero crossing), then $f_1 = 1/t_1$. Measure about 5 periods for better accuracy. How does $f_1$ compare with $f_0$?

**6B.** Measure the exponential decay rate $\gamma$ of the transient amplitudes. $\gamma$ is defined as the reciprocal of the exponential decay time of the envelope function. You may do it using the cursors by measuring the amplitude of the signal at two different times as it falls off and then, assuming the exponential fall-off given (boxed formula on page 3.3), calculate gamma.

**6C.** Theoretically, we expect $Q = \omega_0/(2\gamma)$, where $\omega_0 = 2\pi f_0$. How does this value compare with your measurement? (If there is a large discrepancy, find and correct it.)