Notes on RL, RC circuits and Phasers

Let's now study how the self-inductance affect the circuit behavior.

For that let's start by studying the RL circuit. Even though the emf is induced through the circuit, in our schematic representation we lump the inductance into one part of the circuit.

\[ V - L \frac{dI}{dt} = IR \]

This is part the induced emf.

Here V is the input and therefore it is better to rewrite the latter Eq as

\[ L \frac{dI}{dt} + RI = V \]
The "standard" way to solve a linear differential equation is to look for a solution for the homogeneous Eq. plus a "particular" solution.

The homogeneous solution in this case is

\[
\frac{LdI}{dt} = -RI
\]

\[
\frac{dI}{dt} = -\frac{R}{L} \cdot e
\]

\[
\ln\left(\frac{I}{I_0}\right) = -\frac{R}{L} t
\]

\[
I = I_0 e^{-\frac{R}{L} t}
\]

This is some time called the "natural response".

Now we have to look for a particular solution and it depends on \( V \).

Let's just consider the case where

\[
V = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}
\]

For \( t > 0 \) one can check that the solution \( I_p = I_0 \) is a solution if

\[
I_0 R = V_0 \\
I_0 = \frac{V_0}{R}
\]
The total solution is the sum of the homogeneous and particular solutions:

\[ I(t) = \frac{V_0}{R} + I_p e^{-\frac{t}{\tau}} \]

Since at \( t = 0 \) \( I = 0 \), then \[ I_p = -\frac{V_0}{R} \]

\[ I(t) = \frac{V_0}{R} \left(1 - e^{-\frac{t}{\tau}} \right) \]

The quantity \( \tau = L/R \) is called the time constant. It tells you how long the current takes to reach a substantial value (roughly two-thirds) of its final value.

Had there been no inductance, the current would have jumped immediately to \( V_0/R \).

In the presence of \( L \), the current changes smoothly and continuously.
We want however to study more interesting V's than just a DC. For a general input V(t)
there are a limited set of analytic solutions to the differential equation. However a common V(t) = \sin(\omega t), (-\omega t)
is not only very useful itself but also can be used to build up any arbitrary V(t) using a Fourier sum.

Let's then study an AC power supply.

\[ V(t) = \begin{cases} 0 & t < 0 \\ \left( V_0 \cos(\omega t) \right) & t \geq 0 \end{cases} \]

We already solved the homogeneous solution, we only need the particular solution

\[ L \frac{dI}{dt} + RI = V_0 \cos(\omega t) \]

Let's assume \( I = I_p \cos(\omega t + \phi) \)

Then

\[ -L \omega I_p \sin(\omega t + \phi) \cos(\omega t) + I_p \omega \cos^2(\omega t) = V_0 \cos(\omega t) \]

Collecting terms we have
\[ \cos(\omega t) \left[ R_1 P_0 \cos \phi - 2 \int P_0 \omega \sin \phi \right] - V_0 \\
+ \sin(\omega t) \left[ \omega L_1 P_0 \cos \phi - R_1 P_0 \sin \phi \right] = 0 \]

Using the linear independence of \( \sin \phi \) and \( \cos \phi \):

\[ I_P (\cos \phi - \omega \sin \omega t) = -V_0 \]

\[ \text{Eq. 0: } - \omega L_1 I_P = -V_0 \]
\[ I_P = \frac{V_0}{\omega L_1 - \omega L_1 \omega \cos \omega t} \]

\[ \sin \phi = \frac{-\omega L_1}{\sqrt{\omega^2 L_1^2 + R^2}} \quad \cos \phi = \frac{R}{\sqrt{\omega^2 L_1^2 + R^2}} \]

So:

\[ I_{P0} = \frac{V_0 \sqrt{\omega^2 L_1^2 + R^2}}{R^2 + \omega^2 L_1^2} = \frac{V_0}{\sqrt{R^2 + \omega^2 L_1^2}} \]

Finally:

\[ I_P(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L_1^2}} \cos(\omega t + \theta) \quad \text{for} \quad \theta = -\omega L_1 \]

The total solution is:

\[ I(t) = I_0 e^{-\frac{R}{L_1} t} + \frac{V_0}{\sqrt{R^2 + \omega^2 L_1^2}} \cos(\omega t + \theta) \]

Since \( I(0) = 0 \) then:

\[ I_H = -\frac{V_0 \cos \theta}{\sqrt{R^2 + \omega^2 L_1^2}} = -\frac{V_0 R}{\sqrt{R^2 + \omega^2 L_1^2}} \]
\[ I(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \left( \cos(\omega t + \phi) + \frac{B}{\sqrt{R^2 + \omega^2 L^2}} e^{\frac{-R}{2L} t} \right) \]

This is driven

\[ \downarrow \]

Steady part

This is transient part which dies away

Note that as \( w \to 0 \), \( I \to \frac{V_0}{R} \) (like DC voltage)

But when \( w \to 0 \), \( I \to 0 \)

This means that the high frequency components of a Fourier series will be greatly reduced.

This is why this circuit is called a low-pass filter.

A better way to approach this problem is if we use complex quantities

\[ e^{j \omega t} = \cos(\omega t) + j \sin(\omega t) \]

We at the end have to connect the complex quantities to real world quantities but the algebra is much simpler.
Let's look now for solutions of $V = V_0 e^{i\omega t}$

$V$ is complex and we try a solution

$I = I_0 e^{i\omega t}$

$\tilde{I} \text{Im} e^{i\omega t} + \tilde{I} R e^{i\omega t} = V_0 e^{i\omega t}$

$(\text{cwl + R}) \tilde{I} e^{i\omega t} = V_0 e^{i\omega t}$

so $\tilde{I} = \frac{V_0}{\text{cwl + R}}$

This looks like a simple DC circuit with the reactances replaced by impedances $Z$ while $R$ is complex.

$Z_R = R$ for a pure resistor

$Z_L = \text{cwl}$ for a pure inductor

\[ (\text{since } \frac{d}{dt} = i\omega) \]

Therefore instead of solving a differential equation we solved an algebraic eq

for the complex amplitudes $V$ and $I$.\]
Now let's try to connect the algebraic solution with our real case \( \tilde{V}(t) = V_0 \cos(\omega t) \) which were the one we wanted to solve.

The latter corresponds to \( \tilde{V}_0 = V_0 \) and taking the real part of \( e^{i \omega t} \), since the differential equation is linear we should also take the linear part of \( i e^{i \omega t} \) to find the solution for \( V_0 \cos(\omega t) \).

\[
\text{Re} \left( i e^{i \omega t} \right) = \text{Re} \left( \frac{V_0 e^{i \omega t}}{\omega L + i R} \right)
\]

\[
= \tilde{V}_0 \text{ Re} \left( \frac{e^{i \omega t}}{(\omega L + i R)(-\omega L + i R)} \right)
\]

\[
= \tilde{V}_0 \text{ Re} \left( \frac{e^{i \omega t}}{\omega^2 L^2 + R^2} \right)
\]

\[
= \tilde{V}_0 \left[ \frac{\cos(\omega t)}{\omega^2 L^2 + R^2} \right] \quad \cos \phi = \frac{-\omega L}{R}
\]

This is the same solution we found by solving the differential Eq.
Note this is easily adaptable to
\[ V_0 \sin(\omega t + \phi) \quad \text{or} \quad V_0 \cos(\omega t + \phi) \]

For example \( V(t) = V_0 \sin(\omega t) \)

Then \( \dot{V}_0 = V_0 \) and I must take the
imaginary part

\[
\text{Im} \left( \vec{I} e^{\omega t} \right) = V_0 \text{Im} \left( \frac{e^{\omega t}}{R + j\omega L} \right) = V_0 \text{Im} \left[ \frac{e^{\omega t}}{R^2 + \omega^2 L^2} \right]
\]

\[
= V_0 \left( \frac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2 L^2} \right)
\]

\[
= \frac{V_0}{\sqrt{R^2 + \omega^2 L^2} \cdot \cos \theta} \quad \text{where} \quad \cos \theta = \frac{-\omega L}{R}
\]
Impedance of capacitors

\[ q = CV \]

\[ \frac{dq}{dt} = C \frac{dv}{dt} \quad \frac{dQ}{dt} = I \quad \text{so we have} \]

\[ I = C \frac{dv}{dt} \quad \text{assuming } V(t) = V_c, \text{ or } \quad I = C \omega V_c \]

\[ \text{so } Z_c = \frac{1}{C \omega} \quad \text{or } \quad \frac{-i}{\omega C} \]

Let's solve now an RC circuit

Let's look again for the cases

\[ V(t) = \begin{cases} 0 & t \leq 0 \\ V_0 & t > 0 \end{cases} \]

\[ V(t) = \begin{cases} 0 & t \leq 0 \\ V_0 \sin(\omega t) & t > 0 \end{cases} \]
DC case

\[ V = IR + \frac{Q}{C} \]

\[ \frac{dV}{dt} - \frac{dQ}{dt} R - \frac{I}{C} = 0 \]

Differentiate to get eq for \( I \)

\[ R \frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt} \]

Homogeneous solution

\[ I(t) = I_0 e^{-t/RC} \]

Note: Initial conditions for capacitors

Currents can change instantly, but voltage can not.

Particular solution: Since \( \frac{dV}{dt} = 0 \) it is the same as homogeneous solution

\[ I(t) = I_0 e^{-t/RC} \]

To match for the initial conditions we have

\[ V_0 = I_0 R + \frac{Q(0^+)}{C} \]

\[ Q(0) = 0 \]

\[ I_0 = \frac{V_0}{R} \]

\[ I(t) = \frac{V_0}{R} e^{-t/RC} \]

\[ Q(t) = \int_{0}^{t} I(t) dt = V_0 C (1 - e^{-t/RC}) \]
RC is now the characteristic time instead of \( \frac{1}{C} \)

\[ V - \frac{RI}{C} = 1 \Rightarrow I = 0 \]

\[ i = \frac{\bar{V}}{\left( R + \frac{1}{jwC} \right)} = \bar{V} \frac{wC}{1 + w^2R^2C^2} = \bar{V} \cos(wt - \omega RC) \]

For \( V_0 \sin(wt) \), \( \bar{V} = V_0 \) and we need the imaginary part of \( i \)

\[ \text{Im}(\bar{i} e^{jwt}) = V_0 \frac{wC}{1 + w^2R^2C^2} \left[ \cos(wt) + \sin(wt) \omega RC \right] \]
\[ I(t) = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 RC^2}} \cos(\omega t + \phi) \]

\[ \cos \phi = -\frac{1}{\omega RC} \]

Note the current is 90° shift in phase.

As \( \omega \rightarrow 0 \), \( I \rightarrow 0 \)

\[ \omega \rightarrow \infty \quad I_{\text{max}} \rightarrow \frac{V_0}{R} \]

Capacitor has no effect.

So this is a high pass filter.

To deal with the initial conditions, we have to superimpose the particular and homogeneous solution.

For \( t = 0^+ \)

\[ V(t) = I_0 R + \frac{q(0)}{C} \]

\[ 0 \]

\[ I_0 = 0 \]

\[ I(t) = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 RC^2}} \cos(\omega t + \phi) + I_h e^{-\frac{t}{RC}} \]
\[ I(t) = -\frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2}} \cos \omega t \]

\[ V_c(t) = \frac{C(t) - \int_0^t I(t) dt}{C} \]

\[ V_c(t) = \frac{V_0}{\sqrt{1 + \omega^2 R^2}} \left( \sin(\omega t + \phi) + \frac{wRC \cos \phi e^{-t/RC}}{\sqrt{1 + \omega^2 R^2}} \right) \]

\[ wRC \cos \phi = \frac{wRC}{\sqrt{1 + \omega^2 R^2}} \]

\[ -\sin \phi \]

\[ V_c(t) = \frac{V_0}{\sqrt{1 + \omega^2 R^2}} \left( \sin(\omega t + \phi) - \sin \phi e^{-t/RC} \right) \]

In general, we only care about the amplitude, Im\(X(t)\)