Lecture 6

Let's talk about power supplies.

Power supply is a device which maintains a constant potential difference.

It is represented by \[ \frac{+}{-} \]

Let's connect it to a resistance.

\[ V = V_B - V_A \]

What is the direction of \( E \)? \( E \) always runs from plus to minus potential.

In the resistor, the electric field is in the direction of the current. But in the power supply, it must also go from plus to minus.

So, some kind of pump mechanism forces the current to go inside the supply against the electric field.

In case of batteries, it is chemical energy.
There are consequently two forces involved in driving the current.

The source $F_s$ (which only acts along the battery) and the electrostatic source which communicates the existence of the source across the circuit.

$$f = F_s + E$$

We define as the electromotive force or emf($E$) as the integral of $f$ around the circuit.

(Because $\oint E \cdot dl = 0$ for electrostatic fields)

$$E = \oint f \cdot dl = \oint F_s \cdot dl$$

If $R \to \infty$, and therefore $I = 0$ then the voltage we will measure across the battery

$$V = V_a - V_a = E$$

But if $R$ is not infinitely large a current will start to flow. At this point however we can not forget that the battery always has an internal resistance $r_i$ so the current will not only go through $R$ but also $r_i$. 
The voltage that one will measure between Bond A is now going to change

\[ V = IR \]

\[ V = \mathcal{E} - IR \]

\[ \mathcal{E} = I(R_i + R) \]

\[ V < \mathcal{E} \text{ due to the internal resistance} \]

The power dissipated by the resistance is \( I^2R \).

The power dissipated by the internal resistance is \( IR \).

The maximum power that you can get from the battery is

\[ P_{\text{max}} = \frac{\mathcal{E}^2}{2R} \]

The current is when \( R = 0 \) (when you short out the battery). \( I_{\text{max}} = \frac{\mathcal{E}}{R_i} \)

Example: If you have a 9-volt Duracell battery, the \( \mathcal{E} = 9 \) V, \( r_i = 2 \Omega \).

Then \( I_{\text{max}} = 4.5 \) A, and \( P_{\text{max}} \approx 40 \) watts.

If you take the battery and you short it out, it will get really warm.
Now let's take a look

\[ I_3 = I_1 + I_2 \]

Let's ignore internal resistances for the moment.

We know \( V_1, V_2, R_1, R_2, R_3 \)

What is the current across resistors \( R_1, R_2, R_3 \)?

We use the so-called Kirchhoff's rules:

1) \( \sum E \cdot dI = 0 \) Any loop

2) Charge conservation: current that flows in must flow out.

\[ \sum E \cdot dI = (V_2 - V_1) \]

1) Let's take the outer loop
\[ V_1 + I_1 R_1 - I_2 R_2 - V_2 = 0 \]

\[ I_2 R_2 = (V_1 - V_2) + I_1 R_1 \]

\[ I_2 = \frac{(V_1 - V_2) + I_1 R_1}{R_2} \]
And one inner first loop:

\[ V_1 + I_1 R_1 + R_2 (I_1 I_2) = 0 \]

\[ V_1 + I_1 R_1 + R_3 I_1 + R_3 (V_1 - V_2 + I_1 R_1) = 0 \]

\[ R_2 \]

\[ I_1 \left( \frac{R_1 + R_3 + R_3 R_1}{R_2} \right) = \frac{(V_2 - V_1) R_3}{R_2} = V_1 \]

\[ I_1 \left( \frac{R_1 + R_3 (R_2 + R_1)}{R_2} \right) = \frac{V_2 R_3}{R_2} + V_1 \left( \frac{R_3 + R_2}{R_2} \right) \]

\[ I_1 = \frac{V_2 R_3}{R_3 (R_2 + R_1) + R_1 R_2} - \frac{V_1 (R_3 + R_2)}{R_3 (R_2 + R_1) + R_1 R_2} \]

From \( I_1 \), \( V_1 \) can determine \( I_2 \) and \( I_3 \).

Check let's set \( V_2 = 0 \):

\[ V' = \frac{V'}{R} + \frac{V'}{R_3} + \frac{V'}{R_2} \]

\[ \frac{1}{R} = \frac{1}{R_3} + \frac{1}{R_2} \]

\[ V_1 + I_1 R_1 + I_1 R = 0 \]

\[ V_1 + I_1 R_1 + I_1 R_2 + I_1 R_3 R_2 \]

\[ R_3 + R_2 \]
Electromagnetic Induction

1819 Oersted discovered that a steady current produces a steady magnetic field.

A little later Faraday suggested that a steady magnetic field produces a steady current.

But he did experiments and found it not to be so.

I \Rightarrow B \text{ in the solenoid}

When the switch was closed,
Current in I \Rightarrow \text{ no current in } Z

But one day he noticed that as he closed the switch he saw a current in Z, and as he opened the switch he saw a current.
Not a steady magnetic field but a changing magnetic field is what produced the current.

Electromagnetic Induction: Phenomenon by which a changing magnetic field generates a current.

\[ S \]

As the bar is moving down, B inside the loop increases and it generates a current. The direction of the current is such that it will produce a magnetic field which opposes the change in magnetic field.

This is called Lenz's law.

The induced current is clearly a result of a driving force \( \Rightarrow \) induced emf.

emf in batteries is due to chemical energy, emf can be generated due to a changing magnetic field.

\[ \mathcal{E} = IR \]
Faraday also did experiments with different types of loops.

\[ E_1 \frac{dB}{dt} = E_2 \frac{dA}{dt} \]

\[ E_2 = -\frac{d\Phi_B}{dt} \quad \Phi_B = \int_{S_B} \mathbf{B} \cdot d\mathbf{A} \]

Lien's Law

\[ \oint_{S} \mathbf{E} \cdot d\mathbf{A} = -\frac{d}{dt} \left( \int_{S} \mathbf{B} \cdot d\mathbf{A} \right) \]

Faraday's Law

\[ \oint_{S} \mathbf{E} \cdot d\mathbf{A} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{A} \]

Note that \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \) from Amperes Law.

The curl alone is not enough to determine a field. The divergence must be specified. But the divergence of a pure Faraday field (with \( J = 0 \)) satisfies

\[ \nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = \mathbf{E} \]

\[ \Rightarrow \quad \mathbf{D} = \varepsilon_0 \frac{d\mathbf{E}}{dt} \]
Consider a rectangular circuit moving through an uniform field \( \mathbf{B} \) with constant speed \( \mathbf{V} \) (something is pushing it).

\[
\mathbf{E} = \mathbf{B} \times \mathbf{V} = \mathbf{B} \times \frac{d\mathbf{X}}{dt} = \frac{d}{dt} (\mathbf{B} \times \mathbf{X}) = \mathbf{B} \times \mathbf{V} \\
|E| = |B| |V| \\
\text{direction: clockwise}
\]

Let's look again at it by using Lorentz force.

If there is a velocity \( \mathbf{V} \) along \( x^1 \) then there is a force \( \mathbf{F} = \mathbf{F} \times \mathbf{B} \) in the \( x^1 \) direction (negative charges feel the force in the \( -x^1 \) direction).

But charge can flow across the circuit \( \mathbf{F} \times \mathbf{B} \) has different effects:

- \( d\mathbf{X} \times \mathbf{B} \) pushes charge to "sides" at \( x^1 = 0 \) to no \( \mathbf{E} \cdot d\mathbf{X} \)
- \( d\mathbf{X} \times \mathbf{B} = 0 \) (outside of \( \mathbf{B} \) region) no \( \mathbf{E} \cdot d\mathbf{X} \)
\[ \oint \mathbf{E} \cdot d\mathbf{l} = \int \nabla \times \mathbf{B} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{l} \]

\[ E = VB \text{E} \] Similar to what we got using Faraday's Law

Note that if circuit wasn't closed, charge would build up creating a $\mathbf{E}$ which just balance $\nabla \times \mathbf{B}$

* The jumping ring

When you close the circuit there is a change in magnetic flux through the ring which generates a current.
According to Lenz's law, the current is opposite to the current in the solenoid. Opposite currents repel.

Another example:

Consider a solenoid with changing current:

- \( B = 0 \) outside.
- \( B = \mu_0 I \) inside.

The induced electric field is along \( \hat{z} \), so we can use an "Amperian" loop to calculate \( \vec{E} \):

- \( \oint (\vec{E} \cdot d\vec{l}) = E \cdot 2\pi r \).
- \( \oint B \cdot d\vec{A} = \mu_0 I \).

\[
\frac{d\vec{B}}{dt} = \frac{n \mu_0 I}{2\pi} \frac{dI}{dt}
\]

\[
\vec{E} = \frac{n \mu_0 I}{2\pi} \frac{dI}{dt} \hat{z} = \frac{\mu_0 n I^2}{2\pi} \frac{dI}{dt} \hat{z}.
\]
Note that we have $E$ even in the region where there is not $B$.

Inside

$$\frac{d}{dt} \int \vec{E} \cdot d\vec{A} = \nabla \times \vec{H} \cdot d\vec{A}$$

$$\vec{E} = \nabla \times \vec{H} \cdot \hat{\mathbf{r}}$$

Here we can see clearly the analogy between $2\pi s$ and $\frac{d}{dt}$.

Note that $\vec{E}_{\text{inside}}$ is different than $\vec{E}_{\text{outside}}$ since

$$\nabla \times \vec{E}_{\text{out}} = 0$$

$$\nabla \times \vec{E}_{\text{in}} = -\frac{\partial \vec{B}}{\partial t}$$

This is even though $\vec{E}_{\text{out}}$ also "curls" around the solenoid.
More on motional emf

![Diagram]

Last class we discussed the EMF generated in the circuit shown above.

Q: Where does the energy come from? Does the $B$ field do the work?

First let's examine the forces on the loop.

\[ F = I \int \vec{dl} \times \vec{B} \]

Leg ad \[ F = I \int_a^d \vec{dl} \times \vec{B} = -I \vec{B} \hat{x} \]

Leg dc \[ F = I \int_c^d \vec{dl} \times \vec{B} = I \vec{B} \hat{y} \]

Leg cb \[ F = 0 \quad (B = 0) \]

Leg ba \[ F = I \int_b^a \vec{dl} \times \vec{B} = -I \vec{B} \hat{y} \]

\[ F_{\text{total}} = -I \vec{B} \hat{x} \]

This just opposite to $\vec{V}$ so to maintain
I must push loop with force $I\ell B$.

I must do work on the loop.

If I want to calculate the work done on the loop we have to follow it along its trajectory in time, as work is done on it.

Note that on the contrary, EMF is calculated at one instant in time.

Let's calculate the work.

Since current is flowing and loop is moving, the charge's true velocity has two components:

- $\vec{v}$: from loop motion
- $\vec{v}_u$: from current

The magnetic force on a charge made is $q(\vec{v} \times \vec{B})$ which is $-\vec{B}$ and $+\vec{w}$

$\vec{f}_w = 0$: Magnetic forces do not do work

$\vec{f}_{mag} = q\left(\vec{v} \times \vec{B} + \vec{v}_u \times \vec{B}\right)$
Since the charge is moving with constant velocity along \( \hat{\mathbf{x}} \),

\[ \vec{F}_{\text{mill}} = q\mathbf{v} \times \hat{\mathbf{B}} \hat{\mathbf{x}} \]

\( \vec{F}_{\text{mill}} \) can do work

\[ W = \int \vec{F}_{\text{mill}} \cdot d\vec{r} = \]

\[ W = q \mathbf{v} \cdot \hat{\mathbf{B}} \cdot L \cdot \hat{\mathbf{x}} = q B_0 V \frac{u}{u} \]

\[ W/q = \mathcal{E} \quad \text{work per unit charge is exactly equal to emf} \]

"Push force is " transmitted" to moving charge by mag force"

Analoge "Push force do not do work but allow a horizontally push to be used for vertical motion (against gravity)."
Mutual and Self-Inductance

Now that we have Faraday's Law we need to develop a way to incorporate induction.

If you have a circuit and you run a current through the circuit, you create some magnetic fields, and if the currents are changing there will be an emf in the circuit that fights the change.

To account for that we put in inductor elements in the circuit.

Let's first find a quantitative way to describe how a changing field in circuit (1) affect another circuit (2).

Consider two loops (1 and 2) and a current $I_2$ which flows in loop 1. It creates a magnetic field which is typically not zero inside loop 2.
We can use Biot-Savart law to calculate $\vec{B}_i(r)$

$$\vec{B}_i(r) = \frac{4\pi}{2} \int \frac{\vec{r} \times \vec{A}}{r^2} \, d\vec{l}_i$$

and points from $d\vec{l}_i$ to $\vec{r}$

This can be a complicated function but the important point is that

$$\vec{B}_i(r) \cdot d\vec{l}_i$$

If we calculate the flux on 2 due to 1 then we have

$$\Phi_2 = \int \vec{B}_i \cdot d\vec{r}$$

If $\vec{B}_i$ changes in time, the emf in loop 2 is given by Faraday's Law.

$$E_2 = \frac{d}{dt} \Phi_2 = -\int \frac{d}{dt} \vec{B}_i \cdot d\vec{r}$$

But since $\frac{d\vec{B}_i}{dt} = \frac{d\vec{l}_i}{dt}$ (we are not allowing the shape or relative orientation to change)

then $E_2 = \int \frac{d}{dt} \vec{B}_i \cdot d\vec{r} = -\frac{d}{dt} \Phi_2 = \frac{d}{dt} \frac{\Phi_2}{I}$