MAXWELL'S CORRECTIONS TO AMPERE'S LAW

So far we have the following laws of electrodynamics:

(i) \( \overrightarrow{E} \cdot \overrightarrow{d\alpha} = \frac{Q_{enc}}{\varepsilon_0} \) Gauss's Law

(ii) \( \nabla \times \overrightarrow{B} = 0 \) \( \int_{S} \overrightarrow{B} \cdot d\alpha = 0 \)

(iii) \( \nabla \times \overrightarrow{E} = \mu_0 \overrightarrow{J} \) \( \int_{S} \overrightarrow{E} \cdot d\alpha = \mu_0 I_{enc} \) Ampere's Law

and

(iv) \( \frac{\partial \overrightarrow{E}}{\partial t} = -\frac{\partial \overrightarrow{B}}{\partial t} \) or \( \int_{C} \overrightarrow{E} \cdot dl = -\frac{d}{dt} \int_{S} \overrightarrow{D} \cdot d\alpha \) Faraday's Law

However, there is an inconsistency there:

\( \nabla \cdot (\overrightarrow{D} \times \overrightarrow{B}) = \nabla \cdot (\overrightarrow{D} \times \frac{\partial \overrightarrow{B}}{\partial t}) = -\frac{\partial}{\partial t} \nabla \cdot \overrightarrow{B} = 0 \)

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OK

But \( \nabla \cdot (\overrightarrow{D} \times \frac{\partial \overrightarrow{B}}{\partial t}) = \nabla \cdot (\mu_0 \overrightarrow{J} \times \frac{\partial \overrightarrow{B}}{\partial t}) = \mu_0 \left( \frac{\partial}{\partial t} \right) \overrightarrow{B} \) is not zero necessarily.
What is the problem

Consider a circular plate capacitor

![Diagram of a circular plate capacitor]

we know that inside the capacitor

\[ E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\pi \varepsilon_0 A} = \frac{Q}{\pi R^2 \varepsilon_0} \]

Imagine that we are charging the capacitor.

Then there is a \( q \) which is building up as a function of time. I therefore have a changing electric field

\[ \frac{dE}{dt} = \frac{dq}{dt} \cdot \frac{1}{\varepsilon_0} = \frac{I}{\pi R^2 \varepsilon_0} \]

Let's now calculate the magnetic field of this system.
If I take a point \( P_1 \) at a distance \( s \) from the wire, and if you are far away from the capacitor, one expects that we can calculate \( B \) using Ampere's law due to the symmetry of my system.

If we calculate \( B \) at a point \( P_2 \) but now inside the capacitor, how can we calculate the magnetic field at \( P_2 \)?

Biot-Savart could handle it but it would be very difficult since there is a current in the wire but also going up on the plates.

How about Ampere?

The system has cylindrical symmetry so I can use a closed loop with radius \( r \).

First, let's look at \( P_1 \).
\[ \oint \mathbf{B} \cdot d\mathbf{E} = B(2\pi s) = \mu_0 I \]

\[ B = \frac{\mu_0 I}{2\pi s} \]

How about \( P_2 \)

\[ \oint \mathbf{B} \cdot d\mathbf{E} = 0 \quad B(2\pi s) = 0 \quad B = 0 \quad ?? \]

Makes no sense

Even worse

Let's look again at point \( P_1 \) and instead of a flat surface use

\[ \oint \]
for the surface however

\[ \oint B \cdot d\ell = 0 \quad \text{since there is no current through it} \]

So there is something wrong

Any open surface should give the same answer.

Maxwell realized that and noticed that the only special about points between the capacitor is that there is a changing electric field.

He thought then that if according to Faraday's Law varying magnetic fields give rise to an electric field, it might be the case that varying electric fields can give rise to a magnetic field.

So Maxwell suggested that we need to add a term which contains the derivative of the electric flux to Ampère's Law.

\[ \oint B \cdot d\ell = M_0 I + \nabla \times \mathbf{E} \cdot d\mathbf{a} \]

\[ \frac{d}{dt} \]

Displacement current
Let's revisit our problem with this extra term:

Some answer since there is no $d\phi$ in loop

\[ \oint \mathbf{B} \cdot d\mathbf{S} = \mu_0 \mathbf{E}_0 \int \mathbf{E} \cdot d\mathbf{a} \]

\[ = \frac{\mu_0 \mathbf{E}_0}{2}\pi R^2 \]

\[ \Rightarrow \text{Good some answer than before.} \]

Now it does not matter where you take the flat surface or the "bag" surface.
Let's now go one step ahead and calculate the magnetic field inside the capacitor.

\[
\begin{align*}
\int B \cdot dl &= N\alpha I \mu_0 B_0 \int_{s}^{\infty} \frac{dx}{\sqrt{x^2 - a^2}} \\
&= \frac{N\alpha B_0 I}{\alpha a} \left( \pi a^2 \right)
\end{align*}
\]

For \( s < R \)

\[
B(2\pi s) = \frac{N\alpha B_0 I}{\alpha a} \left( \pi a^2 \right)
\]

\[
B = \frac{M_s}{2\pi} \frac{s}{R^2}
\]

\[
\vec{B} = \vec{B}_0
\]

For \( s > R \)

\[
B(2\pi s) = \frac{\mu_0 c_0 I}{\alpha a}
\]

\[
B = \frac{M_s I}{2\pi s}
\]
The peak and discontinuous derivative of \( B \) at \( s=r \) is artificial since we have neglected fringe effects which can not be neglected or we approach \( s=r \) but anyway for points far from the edges the behavior is correct.

Let's write now the corrected Ampère's law in differential form

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\mu_0}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_0
\]

Note that this addition makes electro-magnetic waves possible.

For \( t=0 \), \( \mathbf{J}=0 \)

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}
\]

and

\[
\nabla \times \left( \frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t} \left( \nabla \times \mathbf{E} \right) = -\frac{\partial}{\partial t} \left( \nabla \times \mathbf{E} \right)
\]

so

\[
\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{E}
\]

This is the wave equation for each component of \( \mathbf{E} \).