Midterm 2 is coming up.

Mon. April 12, 2009

11-11:50AM

G2B47 (our usual lecture room)

Midterm 2 will cover material that we’ve discussed in class on Conservation Laws and the associated local conservation equations that go with them, especially the ones that apply to E&M fields, and associated quantities like the Poynting Vector and the Maxwell Stress Tensor, and finally, EM Waves. These same topics have been the core of the homework assignments, HW6-HW9.

Here are some particular topics that you might find on the exam:

**The Full Versions of Maxwell’s Equations.**
Can you explain them with a picture? Can you go from the integral versions to the partial differential versions and back again? Can you use them?

**Local conservation laws**
Do you understand how they work? Can you figure out the units for some new ‘current density’ if we have a new conserved quantity? Will you have trouble if the current density is a tensor? I am looking for evidence that you have a clear understanding of the final form of Maxwell’s Equations. Do you understand how they can be used with the Lorentz Law to show that charge is conserved, that energy and momentum are conserved, but ONLY if you know how to include the energy and momentum stored in the fields and if you understand how the fields might carry energy and momentum out of the region.

**Electromagnetic Plane waves.**
Can you show that they satisfy Maxwell’s Equations in free space and in homogeneous linear matter? How are the k-vector, frequency, and speed of propagation related? How is the speed of propagation related to materials properties?

**Plane wave Boundary Conditions: Reflection and Refraction**
When plane waves encounter boundaries between materials, you can have reflection and refraction. Can you write down and explain the relationships between the different components of k-vector at the interfaces? How does the frequency change at the boundary? What are the relationships between perpendicular and parallel components of the electric and magnetic fields at the boundary? Can you use these relations to solve the algebraic problem necessary to determine reflected and transmitted plane waves?
Problem 1. Foundations: Maxwell's Equations (20 points)

a) (20 points) Write down the integral and differential equations for each of Maxwell's equations, and draw a picture labeled with the relevant fields and any other quantities that seem important that shows how the fields, surfaces and lines you integrate over, etc. for each of your Maxwell's equations are related.
INTEGRAL Eqs.
\[ \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \]  
Closed surf

DIFFERENTIAL Eqs.
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]
\[ \nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \]  
Closed path

PICTURES

\[ \oint \vec{B} \cdot d\vec{A} = 0 \]  
\[ \nabla \times \vec{B} = 0 \]  
C.S.

\[ \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_E}{dt} \]  
\[ \nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \]  
Closed path

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{through}} + \mu_0 \varepsilon_0 \frac{d\Phi_B}{dt} \]
\[ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \]  
\[ \mu_0 I_{\text{through}} \]  
\[ + \mu_0 \varepsilon_0 \frac{d\Phi_B}{dt} \]
Problem 2. Foundations: Local conservation equations (30 points)

a) (5 points) Write down the local conservation equation for electric charge density. Define your symbols. Write down a relationship between the charge crossing a unit area per unit time, and the charge density and velocity of charge motion.

\[ \frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot \mathbf{J} = 0 \]

\[ \mathbf{J} = \rho \mathbf{v} \]

\[ J = \text{charge density} \]

\[ \mathbf{J} = \text{current density} \]
b) (10 points) Write down the local conservation equation for particle and field energy. Be sure to define your symbols. Write an equation that shows how the electromagnetic field energy crossing a unit area per unit time depends upon the electric and magnetic fields.

\[ \frac{\partial}{\partial t}(U_{\text{field}} + U_{\text{particles}}) + \nabla \cdot \vec{S} = 0 \]

\[ U_{\text{field}} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2 \mu_0} B^2 \]  
energy density in E\&B

\[ U_{\text{particles}} = \text{mechanical energy density of particles} \]

\[ \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \]  
is the energy (Area \& see) carried by EM field.
Consider a long straight cylindrical wire of electrical conductivity $\sigma$ and radius $a$ carrying a uniform axial current of current density $J$. Calculate the magnitude and direction of the Poynting vector at the surface of the wire.

$$E = \frac{J}{\sigma} \hat{z}$$

$$B = \mu_0 J \pi \frac{a^2}{(2 \pi a)} \hat{\phi}$$

$$S = \frac{|E||B|}{\mu_0} (-\hat{r})$$

See example 8.1 Griffiths
Problem 2. Foundations: Local conservation equations (continued)

c) (15 points) Write down the local conservation equation for particle and field linear momentum. Be sure to define your symbols. Write an equation that shows how the electromagnetic momentum per unit volume depends upon the electric and magnetic fields. Write down an equation for the electromagnetic linear momentum crossing a unit area per unit time and show how it depends upon the electric and magnetic fields.
\[ \frac{\partial}{\partial t} (\vec{\rho}_{\text{part}} + \vec{\rho}_{\text{fields}}) = \nabla \cdot \vec{T} \]

\( \vec{\rho}_{\text{part}} \) = momentum density in particles
\( \vec{\rho}_{\text{fields}} = e_0 \vec{E} \times \vec{B} \) = mom. density in EM fields

\[ T_{ij} = e_0 \left[ E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right] + \mu_0 \left[ B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right] \]

Tensor for momentum crossing area per sec in EM fields
Briefly explain how you would calculate the total force on all charges in a given volume using the Maxwell stress tensor.

For steady fields \( F = \int \vec{T} \cdot d\vec{a} \)

The surface must include all the charge in consideration and the tensor is calculated using the total fields. A uniformly charged sphere (with radius \( R \) and total charge \( Q \)) is centered at the origin. By considering the semi-infinite volume bounded by the plane \( z = 0 \), calculate the total force on all charges in the “upper” hemisphere. You may find it helpful to determine which terms of the Maxwell stress tensor are important to calculate before evaluating the appropriate integrals. See example 8.2 Griffiths
Problem 3. Plane wave solutions of Maxwell’s

We have regularly discussed plane wave electromagnetic waves in vacuum. The electric and magnetic fields are then given by:

\[
\vec{E} \left( \vec{r} , t \right) = \vec{E}_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t \right) \right]
\]

\[
\vec{B} \left( \vec{r} , t \right) = \vec{B}_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t \right) \right]
\]

\(E_0\) and \(B_0\) are constant vectors that define the maximum size and directions of the electric and magnetic field vectors respectively.

• Plug this pair of plane wave solutions into Maxwell’s Equations in vacuum. Write the four Maxwell’s equations as algebraic relationships between the \(k\)-vector and the constant vectors \(E_0\) and \(B_0\).
\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

Then become:
\[ i \mathbf{E} \cdot \mathbf{E}_0 = 0 \]
\[ i \mathbf{B} \cdot \mathbf{B}_0 = 0 \]
\[ i \mathbf{E} \times \mathbf{E}_0 = i \omega \mathbf{B}_0 \]
\[ i \mathbf{B} \times \mathbf{B}_0 = -i \omega \mu_0 \varepsilon_0 \mathbf{E}_0 \]
• What is the meaning of the direction of the $k$-vector?
What is the meaning of the magnitude of the $k$-vector.

The $k$-vector points in the direction of wave propagation. The magnitude is related to wavelength via

$$|k| = \frac{2\pi}{\lambda}$$
Explain how you know that these electromagnetic wave solutions are describing transverse waves. In particular, what is the directional relationship of the $k$-vector and the constant vectors $E_0$ and $B_0$?

Transverse waves have $E \perp B \perp$ to direction of propagation i.e. $\perp$ to $k$. We know these waves must be transverse as:

$$k \cdot E_0 = k \cdot B_0 = 0 \text{ i.e. } k \perp E_0 \& k \perp B_0$$
$$k \times E_0 = i\omega B_0 \text{ also shows } E \perp B.$$
How are the magnitudes of $k$ and $\omega$ related to the speed of propagation of the wave? How is this

$$|k| = \frac{2\pi}{\lambda} \quad \& \quad \omega = \frac{2\pi}{\text{period}}$$

so

$$\frac{\omega}{k} = \frac{C}{k} \rightleftharpoons \text{speed of propagation}$$

Speed in vacuum is

$$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$
Problem 4 Reflection, Transmission of EM waves

(i) What are the general boundary conditions (in terms of $E, B, D, H$) at an interface between any two media?

\[ D_1^\perp - D_2^\perp = \sigma \quad \sigma: \text{ surface charge} \]
\[ B_1^\perp - B_2^\perp = 0 \]
\[ E_1^\parallel - E_2^\parallel = 0 \]
\[ H_1^\parallel - H_2^\parallel = K \times \hat{n} \]

K: surface current
(ii) A plane wave in vacuum is incident on a planar surface of a non-conducting, uncharged linear isotropic material. The wave is incident at angle $\theta_I$ from the surface normal, the transmitted wave has angle $\theta_T$ from the surface normal, and the magnetic field of the incident wave is parallel to the surface. With the aid of a diagram give the boundary conditions on the electric field amplitudes of the incident, reflected and transmitted waves ($E_{0I}$, $E_{0R}$ and $E_{0T}$) in terms of $\theta_I, \theta_T, \varepsilon$ and $\mu$.

See Fig. 9.15 and Eq. 9.102-9.104 Griffiths
What is the relation between $\theta$ and $\phi$ (in terms of $\varepsilon$ and $\mu$ only)?

They are related via Snell law

$$\sin \theta = \frac{c}{\sqrt{\mu \varepsilon}} \sin \theta_T$$
Derive the transmitted wave amplitude, $E_0T$, in terms of the incident wave amplitude and $\theta, \phi, \varepsilon$ and $\mu$.

Eq. 9.105-9.109 Griffiths