Electromagnetic Waves

1. Review of waves

what is a wave?
⇒ It is a disturbance of a continuous medium that propagates with a fixed shape at a constant velocity

\[ f(z, t) = f(z-vt, 0) = g(z-vt) \]

The displacement at a point \( z \) at time \( t \) is the same as the displacement at a distance \( vt \) to the left (i.e. at \( z-vt \)) back at \( t=0 \)

• Wave equation:
  \[ \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \]

This admits solutions:
  \[ f(z, t) = g(z-vt) + h(z+vt) \]

  waves propagating in the \( +z \) direction with velocity \( v \)
  waves propagating in the \( -z \) direction with velocity \( v \)

• Sinusoidal waves

Are waves of the form
  \[ f(z, t) = A \cos \left( \frac{2\pi}{\lambda} (z-vt) + \phi \right) \]

\( A \): amplitude. It is positive and represents the maximum displacement from equilibrium
\( \phi \): phase
\( \delta \): phase constant

At \( z = v t - \frac{\delta}{k} \), the phase is zero and let's call it "central maximum".
\( \frac{\delta}{k} \) is the distance by which the central maximum is delayed.

\( k \): wave number
\[ k = \frac{2\pi}{\lambda} \quad \lambda \) is the wavelength

When \( z \) advances by \( \frac{2\pi}{k} = \lambda \), the cosine executes one complete cycle.

At a fixed point \( z \), \( f \) vibrates up and down undergoing a full cycle in a period

\[ T = \frac{2\pi}{k v} \]

The frequency \( v \) (number of oscillations per unit time) is

\[ v = \frac{1}{T} = \frac{k v}{2\pi} = \frac{\lambda}{T} \]

The angular frequency
\[ \omega = 2\pi v = k v \] number of radians swept out per unit time

we can rewrite the sinusoidal wave as

\[ f(z,t) = A \cos(k z - \omega t + \delta) \]

**Complex Notation:**
It is convenient to introduce the complex wave function 
\[ f(x,t) = \text{Re} \left[ A e^{i(kx - \omega t + \phi)} \right] \]

\[ \tilde{A} = A e^{i\phi} \]

\[ f(x,t) = \text{Re} \left[ \tilde{A} e^{i(kx - \omega t)} \right] \]

Simpler to manipulate.

**Linear combination:**

Any wave can be expressed as a linear combination of sinusoidal waves:

\[ \tilde{f}(x,t) = \int \tilde{A}(k) e^{i(kx - \omega t)} \, dk \]

\( w \): is a function of \( k \) and negative values of \( k \) mean waves going in the opposite direction.

**Polarization**

- **Transverse waves:** displacement perpendicular to the direction of propagation:
  - waves in a string
- **Longitudinal waves:** displacement along the propagation direction: e.g., sound waves which are just compression waves in air.

We will see next that EM waves are transverse.

Note that for a given propagation there are two possible polarizations:

- **Vertical polarization**
  \[ \tilde{f}_v(x,t) = \tilde{A} e^{i(kx - \omega t)} \]
- **Horizontal polarization**
  \[ \tilde{f}_h(x,t) = \tilde{A} e^{i(kx - \omega t)} \]

In general:

\[ \tilde{f}(x,t) = \tilde{A} e^{i(kx - \omega t)} \hat{\mathbf{n}} \]

\( \hat{\mathbf{n}} \): a vector in the
In general, \( \vec{f}(x,y) = \hat{a} e^{(i(kx - \omega t))} \) \( \hat{a} \) a vector in the xy plane

\( \hat{a} \cdot \hat{z} = 0 \) for transverse waves

\( \hat{a} \) determines the plane of vibration. \( \hat{a} = \cos \theta \hat{x} + \sin \theta \hat{y} \)

Electromagnetic waves

In regions of space where there is no charge or current, Maxwell’s equations read:

\( \text{i} \ \nabla \cdot \vec{E} = 0 \), \( \text{ii} \ \nabla \cdot \vec{B} = 0 \), \( \text{iii} \ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \), \( \text{iv} \ \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \)

They can be decoupled by for example applying \( \nabla \times \) to \( \text{iii} \) and \( \text{iv} \):

\( \nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla (-\frac{\partial \vec{B}}{\partial t}) \)
\[ -\frac{2}{\varepsilon_0} \frac{\partial E}{\partial t} = -\frac{\partial B}{\partial t} \]

\[ \nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \gamma \gamma (\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}) \]

\[ = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) \]

\[ -\gamma \gamma \frac{\partial^2 \vec{B}}{\partial t^2} \]

So

\[ \nabla \times \vec{E} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \cdot \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \frac{1}{v^2} = \frac{1}{c^2} \]

An 

wave moves at the speed of light.